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Note

Proximity, remoteness and minimum degree

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ABSTRACT

The average distance $\overline{\sigma}(v)$ of a vertex v of a connected graph G is the average of the distances between v and all other vertices. The remoteness $\rho(G)$ and proximity $\pi(G)$ of G are defined as $\max_{v\in V(G)} \overline{\sigma}(v)$ and $\min_{v\in V(G)} \overline{\sigma}(v)$, respectively. Zelinka (1968) and, independently, Aouchiche and Hansen (2011) showed that the proximity and remoteness of a connected graph of order n are bounded by approximately $\frac{n}{4}$ and $\frac{n}{2}$, respectively, and that the difference between the remoteness and proximity is bounded by approximately $\frac{n}{4}$. We show that for graphs of minimum degree δ, where $\delta \geq 2$, all three bounds can be improved by a factor of about $\frac{3}{\delta+1}$. Our bounds are sharp except for an additive constant.

1. Introduction

For location problems in a network, it is often desirable to locate a facility in a vertex whose distance to the vertex farthest away is minimal, or in a vertex whose distance to all other vertices on average is small. Usually the former is the best location for an emergency facility which should be reachable from all vertices within a given time, and the latter is the best location for a service facility which should on average be reachable from all vertices within minimal time. Under these assumptions, the suitability of a vertex as a location for an emergency facility is described by its eccentricity, i.e., the distance to a vertex farthest from it, while the suitability of a vertex as a location for a service facility is described by its average distance, i.e., the average of the distances to all other vertices. We denote the average distance of a vertex v in a graph G by $\overline{\sigma}(v, G)$. The vertices of minimum average distance in G are the median vertices of G.

For a vertex v of a connected graph G we define the distance $\sigma(v, G)$ (sometimes also called its transmission) and the average distance $\overline{\sigma}(v, G)$ of v as the sum and the average, respectively, of the distances from v to all other vertices, i.e.,

$$\sigma(v, G) = \sum_{w\in V(G)} d_G(v, w),$$

$$\overline{\sigma}(v, G) = \frac{1}{|V(G)| - 1} \sum_{w\in V(G)} d_G(v, w),$$

where $d_G(v, w)$ denotes the distance between v and w in G, i.e., the minimum number of edges on a path from v to w. The proximity $\pi(G)$ and remoteness $\rho(G)$ of a connected graph G are defined as the smallest and the largest, respectively, of the average distances of all vertices, i.e.,

$$\pi(G) = \min_{v\in V(G)} \overline{\sigma}(v, G), \qquad \rho(G) = \max_{v\in V(G)} \overline{\sigma}(v, G).$$

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The largest and smallest value of the proximity and remoteness of graphs of given order were determined by Zelinka [8] and, independently, by Aouchiche and Hansen [3].

Theorem 1 (*Zelinka [8], Aouchiche, Hansen [3]*)**.** *Let G be a connected graph of order n, where $n \geq 2$. Then*

$$1 \leq \pi(G) \leq \begin{cases} \dfrac{n+1}{4} & \text{if } n \text{ is odd,} \\ \dfrac{n+1}{4} + \dfrac{1}{4(n-1)} & \text{if } n \text{ is even.} \end{cases}$$

The lower bound holds with equality if and only if G has a vertex of degree $n-1$. The upper bound holds with equality if and only if G is a path or a cycle.

$$1 \leq \rho(G) \leq \frac{n}{2}.$$

The lower bound holds with equality if and only if G is a complete graph. The upper bound holds with equality if and only if G is a path.

An upper bound on the difference between the proximity and the remoteness of a graph was given by Aouchiche and Hansen [3].

Theorem 2 (*[3]*)**.** *Let G be a connected graph of order n. Then*

$$\rho(G) - \pi(G) \leq \begin{cases} \dfrac{n-1}{4} & \text{if } n \text{ is odd,} \\ \dfrac{n-1}{4} - \dfrac{1}{4(n-1)} & \text{if } n \text{ is even,} \end{cases}$$

and this boundis sharp.

In this paper we show that the above bounds on the proximity, the remoteness and their difference can be improved by a factor of about $\frac{3}{\delta+1}$ for graphs of minimum degree δ, where $\delta \geq 2$. Our bounds are sharp except for an additive constant.

For further results on the proximity and remoteness of a graph see, for example, [1,3,2,4,6,7].

The notation we use is as follows. For a graph G we denote its vertex set by $V(G)$ and its edge set by $E(G)$. We write $n(G)$ for the number of vertices of G. For a vertex v of G, the set $N_G(v)$ is the neighbourhood of v, i.e., the set of vertices adjacent to v. The degree of a vertex v, denoted by $\deg_G(v)$, is the cardinality $|N_G(v)|$, and $\delta(G)$ is the minimum degree of G. For two vertices v, w of G, their distance, i.e., the minimum length of a $v-w$ path is denoted by $d_G(v, w)$. For a vertex v and a set $A \subseteq V(G)$ we define the distance between v and A as $\min_{a \in A} d_G(v, a)$. The subscript or argument G may be dropped if the graph is understood.

The eccentricity of a vertex v in a graph G is defined as $\max_{w \in V(G)} d_G(v, w)$. The radius $\text{rad}(G)$ of G is the minimum of all eccentricities, and a centre vertex of G is a vertex of eccentricity $\text{rad}(G)$. The kth power of G, denoted by G^k, is the graph with vertex set $V(G)$, where two vertices are adjacent if and only if their distance in G is not more than k. For $A \subseteq V(G)$, we write $G[A]$ for the graph induced by A. A set $A \subseteq V(G)$ is a packing of G if the distance between any two vertices of A is at least three. If $G_1, G_2, \ldots, G_k$ are disjoint graphs, then the sequential sum $G_1 + G_2 + \cdots + G_k$ is the graph obtained from their union by adding an edge from every vertex in G_i to every vertex in G_{i+1} for $i = 1, 2, \ldots, k-1$.

2. Results

We first give bounds on the proximity and remoteness of graphs of given order and minimum degree. For the proof of our bounds, which employs a technique first used in [5], we require a generalisation of the distance of a vertex, the weighted distance. Let G be a connected graph G and $v \in V(G)$. For a weight function $c : V(G) \longrightarrow \mathbb{N}_0$ we define the weighted distance of a vertex v as

$$\sigma_c(v, G) = \sum_{w \in V(G)} c(w) d_G(v, w).$$

We use the convention that for a subset $M \subseteq V(G)$ we write $c(M)$ for $\sum_{v \in M} c(v)$. Lemma 1 generalises the upper bound on $\rho(G)$ in Theorem 1, which is the case $N = n$ and $a = 1$.

Lemma 1. *Let G be a connected graph, a a positive integer, and $c : V(G) \longrightarrow \mathbb{N}_0$ a weight function such that $c(v) \geq a$ for all $v \in V(G)$. Let $N := c(V(G))$. If v_0 is an arbitrary vertex of G, then*

$$\sigma_c(v_0) \leq \frac{(N-a)N}{2a},$$