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Characterization of split graphs with at most four distinct eigenvalues

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ABSTRACT

A graph whose vertex set can be partitioned into the disjoint union of an independent set and a clique is called split graph. A complete split graph is one that all vertices of independent set are adjacent with the vertices of the clique. In this paper, all split graphs with at most four distinct eigenvalues are characterized.

1. Introduction

In this section, we introduce some basic notation and terminology used throughout the paper. All graphs considered are connected simple. Our notation is standard and mainly taken from [2,5,3].

For any two nonadjacent vertices x and y in graph G, we use $G+xy$ to denote the graph obtained from adding a new edge xy to graph G. Similarly, for $e = xy$ in $E(G)$, $G-xy$ represents a new graph obtained from graph G by deleting the edge $e = xy$.

The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2.

A stable set in a graph is a set of vertices no two of which are adjacent. A clique in a graph is a set of mutually adjacent vertices. The maximum size of a stable set is called the independent number and we denote it by $\alpha(G)$. Similarly, the clique number of G is a clique with the maximum size and we denote it by $\omega(G)$. Clearly, a set of vertices S is a clique of a simple graph G if and only if it is a stable set of its complement $\bar{G}$. In particular, $\alpha(G) = \omega(\bar{G})$.

We denote by K_n, C_n and S_n, the complete graph, the cycle graph and the star graph on n vertices, respectively. A n-sun graph is a graph on $2n$ vertices consisting of a central complete graph K_n with an outer ring of n vertices, each of which is joined to both endpoints of the closest outer edge of the central core.

The split graph $G = S(r, t)$ is a graph whose vertex set is $V = S \dot{\cup} Q$ where S, Q denote to the stable set and clique of G, respectively and $|S| = r$, $|Q| = t$. It was shown in [9] that G is split if and only if it does not have an induced subgraph isomorphic to one of the three graphs C_4, C_5 or $2K_2$. One can prove that the complement and every induced subgraph of a split graph is split. A complete split graph is one whose all vertices of stable set are adjacent with the vertices of clique of G. Suppose S and Q be respectively, the set of vertices of stable set and clique of G with r and t vertices. If the split graph G is complete, then $G = \bar{K_r} + K_t$. We denote a complete split graph by $CS(r, t)$, see [4,7].

In [6], the authors characterized all integral signless Laplacian of complete split graphs and in the present work, we determine the spectrum of all complete split graphs. A spectral characterization of families of split graphs, involving its index

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and entries of the principal eigenvectors was given in [1]. We encourage the reader to see the recent paper of Anđelić and Cardoso [1] for current and complete descriptions as well as [8].

Let A denote the adjacency matrix of graph G, then the characteristic polynomial $\chi(G, x)$ of graph G is defined as

$$\chi(G, x) = |A - xI|.$$

The roots of the characteristic polynomial are the eigenvalues of graph G and form the spectrum of G. In the next section, we classify all split graphs with maximum degree $\Delta \leq 4$ and then we compute the full spectrum of complete split graph, generally. Here, we are interested to characterize all split graphs with at most four eigenvalues.

Theorem 1. *Every complete split graph $CS(r, t)$ has four eigenvalues. Further, its spectrum is $\left\{[-1]^{t-1}, [0]^{r-1}, [\lambda]^1, [\mu]^1\right\}$, where*

$$\lambda, \mu = \frac{t - 1 \pm \sqrt{(1-t)^2 + 4tr}}{2}.$$

Proof. It is easy to see that, the adjacency matrix of $CS(r, t)$ is

$$A(CS(r, t)) = \begin{pmatrix} 0_{r\times r} & J_{r\times t} \\ J_{t\times r} & (J - I)_{t\times t} \end{pmatrix}.$$

Hence, the characteristic polynomial of $CS(r, t)$ is

$$\begin{aligned} \chi(CS(r, t)) &= |\lambda I_{t+r} - A| = \lambda^r |\lambda I_t - (J - I)_{t\times t} - \frac{1}{\lambda} J_{t\times r} \times J_{r\times t}| \\ &= \lambda^{r-t} |\lambda^2 I_t - \lambda(J - I)_t - rJ_t| = \lambda^{r-t} |(\lambda^2 + \lambda) I_t - (\lambda + r) J_t| \\ &= \lambda^{r-t} (\lambda + r)^t \chi_{\frac{\lambda^2+\lambda}{\lambda+r}}(J_t) \\ &= \lambda^{r-t} (\lambda + r)^t \left(\frac{\lambda^2 + \lambda}{\lambda + r}\right)^{t-1} \left(\frac{\lambda^2 + \lambda}{\lambda + r} - t\right) \\ &= \lambda^{r-t} \lambda^{t-1} (\lambda + r)^t \left(\frac{\lambda + 1}{\lambda + r}\right)^{t-1} \left(\frac{\lambda^2 + \lambda - t\lambda - tr}{\lambda + r}\right) \\ &= \lambda^{r-1} (\lambda + 1)^{t-1} (\lambda^2 + (1 - t)\lambda - tr). \end{aligned}$$

This completes the proof.

2. Results and discussion

In this section, we classify all split graphs with at most four distinct eigenvalues. First, we introduce an upper bound for the number of vertices of a split graph with respect to its maximum degree Δ. To do this, assume that G is a split graph with $\Delta \leq 4$ and Q denotes to the maximum clique of G. We claim that $|V(G)| \leq 9$. It is easy to prove that the maximum number of vertices of Q is five. If Q has five vertices, then $G \cong K_5$ and so G has only five vertices. If Q has four vertices, since G is connected, according to pigeonhole principle the maximum number of vertices of its stable set is at most four and hence in this case G has exactly eight vertices. Let Q has three vertices. The condition $\Delta \leq 4$ leads us to verify that the maximum number of vertices of stable set is six and all of them are pendant, thus in this case G has nine vertices. Finally, if Q is the complete graph K_2, then the number of vertices of any stable set is at most six and so G has eight vertices. In general, by continuing our method, we have the following result.

Proposition 1. *If G is a split graph with maximum degree Δ, then*

$$|V(G)| \leq \left[(\Delta - 2)^2/4\right] + 2\Delta.$$

Proof. It is easy to see that $|Q| \leq \Delta + 1$. If $|Q| = \Delta + 1$, then $|S| = \phi$ and so G is a complete graph on $\Delta + 1$ vertices. If $|Q| = \Delta$ then for every vertex v in Q, $deg_Q(v) = \Delta - 1$ and so G has at most $|Q| + |Q| = 2\Delta$ vertices. By continuing this method, we can take $|Q| = \Delta - i$ and hence $deg_Q(v) = \Delta - i - 1$. This implies that the maximum number of vertices is

$$|Q| + (i + 1)|Q| = (i + 2)(\Delta - i).$$

Consider the real function $f(x) = x\Delta - x^2 + 2\Delta - 2x$ where $0 \leq x \leq \Delta - 2$. It is clear that

$$f'(x) = -2x + \Delta - 2 = 0 \Leftrightarrow x = (\Delta - 2)/2.$$

On the other hand, $f(0) = f(\Delta - 2) = 2\Delta$ and $f((\Delta - 2)/2) = (\Delta - 2)^2/4 + 2\Delta \geq 2\Delta$. Hence, the maximum value of this function holds at point $(\Delta - 2)/2$ and so G has at most $f((\Delta - 2)/2) = [(\Delta - 2)^2/4] + 2\Delta$ vertices. This completes the proof.