Discrete Applied Mathematics 184 (2015) 243–247

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

(3, 1)-Choosability of toroidal graphs with some forbidden short cycles☆

Yubo Jing, Yingqian Wang*

College of Mathematics, Physics and Information Engineering, Zhejiang Normal University, Jinhua, 321004, China

ARTICLE INFO

Article history:
Received 13 January 2014
Received in revised form 4 September 2014
Accepted 18 October 2014
Available online 13 November 2014

Keywords:
Toroidal graph
Cycle
Improper choosability

ABSTRACT

Lih et al. showed that every planar graph without 4-cycles or i-cycles for some $i \in \{5, 6, 7\}$ is (3, 1)-choosable. Dong and Xu showed that every toroidal graph without 4-cycles or 6-cycles is (3, 1)-choosable. In this paper, we show that every toroidal graph without 4-cycles or i-cycles for some $i \in \{5, 7\}$ is also (3, 1)-choosable.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. A *toroidal* graph G is a graph drawn on the torus without crossing edges. We use $V(G)$, $E(G)$, $F(G)$ and $\delta(G)$ to denote the vertex set, edge set, face set and minimum degree of G, respectively.

A *list assignment* L of G assigns a list $L(v)$ of available colors for every $v \in V(G)$. A k-*list assignment* satisfies that $|L(v)| \geq k$ for any vertex v. An L-*coloring* ϕ of a graph G is a coloring of G such that $\phi(v) \in L(v)$ for every $v \in V(G)$, and $\phi(x) \neq \phi(y)$ whenever $xy \in E(G)$. G is L-*colorable* if it admits an L-coloring. Call G k-*choosable* if it is L-colorable for every k-list assignment L. Let d be a non-negative integer. An (L, d)-coloring is a mapping ϕ that assigns a color $\phi(v) \in L(v)$ to each vertex $v \in V(G)$ such that at most d neighbors of v receive color $\phi(v)$. A graph G is called (k, d)-choosable if it admits an (L, d)-coloring for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$. Clearly, the (k, d)-choosability generalizes the classical k-choosability.

Eaton and Hull [4] and Škrekovski [6] independently showed that every planar graph is (3, 2)-choosable. Škrekovski [7] showed that every planar graph without 3-cycles is (3, 1)-choosable. Later, Lih et al. [5] showed that every planar graph without 4-cycles or i-cycles for some $i \in \{5, 6, 7\}$ is (3, 1)-choosable. Dong and Xu [2] extended this result for $i \in \{8, 9\}$. Recently Wang and Xu [8] improved these results by proving that every planar graph without 4-cycles is (3, 1)-choosable. Very recently, Chen and Raspaud [1] further improved this result by proving that every planar graph without adjacent 4^--cycles is (3, 1)-choosable.

As for improper choosability of toroidal graphs, Xu and Zhang [10] showed that every toroidal graph without adjacent triangles is (4, 1)-choosable. Xu and Yu [9] proved that every toroidal graph with neither adjacent triangles nor 6-cycles is (3, 1)-choosable if it further has no l-cycles for some $l \in \{5, 7\}$. Dong and Xu [3] showed that every toroidal graphs with neither 4-cycles nor 6-cycles is (3, 1)-choosable. In this paper, we show the following result.

☆ Supported by NSFC No. 11271335.
* Corresponding author.
E-mail address: yqwang@zjnu.cn (Y. Wang).

http://dx.doi.org/10.1016/j.dam.2014.10.030
0166-218X/

Theorem 1. *Every toroidal graph G with neither 4-cycles nor k-cycles for some $k \in \{5, 7\}$ is $(3, 1)$-choosable.*

The rest of this section is devoted to some definitions. For $v \in V(G)$, let $d(v)$ and $N(v)$ denote the degree and the neighborhood of $v \in G$, respectively. A vertex of degree k, (resp. at least k, at most k) will be called a k-vertex (resp. k^+-vertex, k^--vertex). A similar notation will be used for cycles and faces, too. In this paper, a triangle is synonymous with a 3-face. A vertex or an edge is called *triangular* if it is incident with a triangle. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [v_1v_2 \cdots v_n]$ if $v_1, v_2, \ldots, v_k$ are the boundary vertices of f in a cyclic order, and say that f is a $(d(v_1), d(v_2), \ldots, d(v_k))$-face. For any $v \in V(G)$, we let $v_1, v_2, \ldots, v_{d(v)}$ denote the neighbors of v in a cyclic order. Let f_i be the face with vv_i and vv_{i+1} as two boundary edges for $i = 1, 2, \ldots, d(v)$, where indices are taken modulo $d(v)$. Let $vv' \in E(G)$. If vv' is non-triangular, then v' is called an *isolated* neighbor of v. If v' is an isolated 3-neighbor of v and v' is triangular, then v' is called a *pendent* neighbor of v. Finally, two cycles or two faces are *adjacent* if they have at least one edge in common; *normally adjacent* if their intersection consists of exactly one edge.

2. Reducibility

Suppose Theorem 1 is false. Let $G = (V, E, F)$ be a counterexample to Theorem 1 with $\sigma(G) = |V| + |E|$ as small as possible. Clearly G is connected. Below are some known structural properties of G.

Lemma 1. (1) $\delta(G) \geq 3$ [5].
(2) *G has no adjacent 3-vertices* [5].
(3) *G has no $(4^-, 4^-, 4^-)$-face* [1].
(4) *A 4-vertex has at most two 3-neighbors* [1].

According to Lemma 1(4), call a 4-vertex *bad* if it has exactly two isolated 3-neighbors; good otherwise. For short, by a *4^b-vertex* (*4^g-vertex*, resp.) we mean a bad (good, resp.) 4-vertex.

Call a vertex v *B-vertex* if v is a 3-vertex or a 4^b-vertex.

Let $\emptyset \neq A \subset V(G)$ and $G' = G - A$. By the minimality, G' admits an $(3, 1)$-coloring ϕ using color from $L(x)$ for every $x \in V(G')$. For convenience, we use the notation $\phi(a)$ to denote the color assigned to $a \in A$ in the process of extending ϕ from G' to G. Note that $\phi(A') = \{\phi(a) | a \in A' \subseteq A\}$ may be a multi-set of colors.

Lemma 2. *G has no adjacent 4^b-vertices.*

Proof. Suppose to the contrary that G has two adjacent 4^b-vertices, say v_1 and v_2. Let $v_i^j, j = 1, 2, 3$, be the remaining three neighbors of v_i, where v_i^1 and v_i^2 are the two isolated 3-neighbors of v_i. Note that neither v_1^3 nor v_2^3 is a 3-vertex, and v_1^3 and v_2^3 may be identical. Let G' be the graph obtained by deleting v_1, v_2 and all their isolated 3-neighbors from G. By the minimality of G, G' admits a $(3, 1)$-coloring ϕ (using color from $L(x)$ for every $x \in V(G')$). First we properly color v_i^j for $i, j = 1, 2$. If $\phi(v_1^1) = \phi(v_1^2)$, then we properly color v_1, then color v_2 with a color from $L(v_2) \setminus \{\phi(v_2^3)\}$, which appears in $\{\phi(v_1), \phi(v_2^1), \phi(v_2^2)\}$ at most once. The same argument works if $\phi(v_2^1) = \phi(v_2^2)$. We may assume that, for $i = 1, 2$, $\phi(v_i^1) \neq \phi(v_i^2)$. Now we first color v_1 with a color from $L(v_1) \setminus \{\phi(v_1^3)\}$, then color v_2 with a color from $L(v_2) \setminus \{\phi(v_1), \phi(v_2^3)\}$, obtaining a $(3, 1)$-coloring of G with respect to L, a contradiction. □

By Lemmas 1 and 2, it is easy to deduce the following lemma.

Lemma 3. *If f is a 3-face in G, then f is a $(B, 4^g, 5^+)$-, $(B, 5^+, 5^+)$-, $(4^g, 4^g, 5^+)$-, $(4^g, 5^+, 5^+)$-, or $(5^+, 5^+, 5^+)$-face.*

Lemma 4. *A 5-vertex in G has at most four B-neighbors, in particular, four 3-neighbors.*

Proof. Suppose to the contrary that all neighbors of v are B-vertices. Let v_i, $i = 1, 2, 3, 4, 5$, be the neighbors of v. We may assume that all neighbors of v are 4^b-vertices, since if some neighbors of v are 3-vertices, then the proof is similar (even easier). Let v_i', v_i'', v_i''' be the three neighbors of v_i other than v, where v_i' and v_i'' are the two 3-vertices defining v_i to be a 4^b-vertex. Deleting $v, v_i, i = 1, 2, 3, 4, 5$, together with their 3-neighbors from G, we obtain a smaller graph G'. By the minimality of G, G' admits a $(3, 1)$-coloring ϕ. To extend ϕ from G' to G, we first properly color v_i', v_i'', for $i = 1, 2, 3, 4, 5$, then color v_i with a color from $L(v_i) \setminus \{\phi(v_i''')\}$, which appears in $\{\phi(v_i'), \phi(v_i'')\}$ at most once. Now we color v with a color from $L(v)$, which appears on the neighbors of v at most once. This gives a $(3, 1)$-coloring of G unless the color on v, say α, is used on v_i and one 3-neighbor of v_i for some $i \in \{1, 2, 3, 4, 5\}$. In that case, we can recolor v_i with a color from $L(v_i) \setminus \{\alpha, \phi(v_i''')\}$, giving a $(3, 1)$-coloring of G with respect to L, a contradiction. □

Lemma 5 (*[8]*)**.** *No 5-vertex in G is incident with*
(1) *Two $(B, 4, 5)$-faces.*
(2) *One $(B, 4, 5)$-face, and has two isolated 3-neighbors.*
(3) *Two $(4^-, 4^-, 5)$-faces, and has one isolated 3-neighbor.*
(4) *One $(5, B, 4)$-face and one $(5, B, 4^+)$-face, and has one isolated 3-neighbor.*

Lemma 6 (*[8]*)**.** *No 6-vertex in G is incident with one $(6, B, 4^g)$-face and two $(6, 4^-, 4^-)$-faces.*