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Note

Degree conditions for graphs to be maximally k-restricted edge connected and super k-restricted edge connected[☆]

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ABSTRACT

For a connected graph $G = (V, E)$, an edge set $S \subseteq E(G)$ is called a k-restricted edge cut of G if $G-S$ is disconnected and every component of $G-S$ has at least k vertices. The k-restricted edge connectivity of G, denoted by $\lambda_k(G)$, is defined as the cardinality of a minimum k-restricted edge cut. For two disjoint vertex subsets X, Y of G, define $[X, Y] = \{xy \in E(G) : x \in X, y \in Y\}$ and define $\xi_k(G) = \min\{|[X, \overline{X}]| : X \subseteq V(G), |X| = k, G[X] \text{ is connected}\}$, where $\overline{X} = V(G) \setminus X$. G is λ_k-optimal if $\lambda_k(G) = \xi_k(G)$. Furthermore, G is super-λ_k if every minimum k-restricted edge cut of G isolates a connected subgraph with order k. The k-restricted edge connectivity is an important index to estimate the reliability of networks. In this paper, some degree conditions for graphs to be maximally k-restricted edge connected and super k-restricted edge connected are given.

1. Introduction and definitions

For graph-theoretical terminology and notation not defined here we follow [1].

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. The order of G is the number of vertices in G, denoted by ν. If vertices u and v are connected in G, the distance between u and v in G, denoted by $d(u, v) = d_G(u, v)$, is the length of a shortest path from u to v in G. If there is no path connecting u and v, we define $d(u, v)$ to be infinite. For each vertex $v \in V(G)$, the open neighborhood $N(v) = N_G(v)$ of v is defined as the set of all vertices adjacent to v, $N_G[v] = N_G(v) \cup \{v\}$ is the closed neighborhood of v, and $d(v) = |N(v)|$ is the degree of v. For $X \subseteq V(G)$, we define $N_X(v) = N_G(v) \cap X$. A cut vertex of a graph G is a vertex v such that $c(G - v) > c(G)$, where $c(G)$ denotes the number of components of G. In particular, a cut vertex of a connected graph is a vertex whose deletion results in a disconnected graph. For two disjoint vertex sets X and Y, let $[X, Y]$ be the set of edges with one endpoint in X and the other one in Y, and let $|[X, Y]|$ denote the cardinality of $[X, Y]$. If $X \subseteq V(G)$, then let $\overline{X} = V(G) \setminus X$, and let $G[X]$ be the subgraph induced by X. We define $\partial(X) = |[X, \overline{X}]|$.

An edge set $S \subseteq E(G)$ is said to be a restricted edge cut if $G - S$ is disconnected and every component of $G - S$ contains at least 2 vertices. The restricted edge connectivity of G, denoted by $\lambda'(G)$, is defined as the cardinality of a minimum restricted edge cut of G. Denote the minimum edge degree by $\xi(G) = \min\{d(u) + d(v) - 2 : uv \in E(G)\}$. A graph G with $\lambda'(G) = \xi(G)$ is said to be λ'-optimal. Furthermore, if every minimum restricted edge cut of G is a set of edges adjacent to a

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certain edge with minimum edge degree in G, then G is said to be super-λ'. Generally, for a connected graph G, an edge set $S \subseteq E(G)$ is called a k-restricted edge cut of G if $G - S$ is disconnected and every component of $G - S$ has at least k vertices. The k-restricted edge connectivity of G, denoted by $\lambda_k(G)$, is defined as the cardinality of a minimum k-restricted edge cut. A minimum k-restricted edge cut is called a λ_k-cut. A connected graph G is called a λ_k-connected graph if $\lambda_k(G)$ exists. Clearly, $\lambda_2(G) = \lambda'(G)$. It is easy to see that if G is λ_k-connected for $k \geq 2$, then G is also λ_{k-1}-connected and $\lambda_{k-1}(G) \leq \lambda_k(G)$. In view of recent studies on k-restricted edge connectivity, it seems that the larger $\lambda_k(G)$ is, the more reliable the network is [6,7,10]. So, we expect $\lambda_k(G)$ to be as large as possible. Clearly, the optimization of $\lambda_k(G)$ requires an upper bound first. For any positive integer k, let $\xi_k(G) = \min\{|[X, \overline{X}]| : X \subseteq V(G),\ |X| = k,\ G[X] \text{ is connected}\}$. A connected graph G is called a maximally k-restricted edge connected graph, in short, a λ_k-optimal graph, if $\lambda_k(G) = \xi_k(G)$. Furthermore, G is called a super k-restricted edge connected graph, in short, a super-λ_k graph, if every λ_k-cut of G isolates a connected subgraph with order k, that is, every λ_k-cut of G is a set of edges adjacent to a certain connected subgraph with order k. Let G be a λ_k-connected graph with $\lambda_k(G) \leq \xi_k(G)$. By definition, if G is a super-λ_k graph, then G must be a λ_k-optimal graph. However, the converse is not true. For example, a cycle of length $\nu(\nu \geq 2k + 2)$ is a λ_k-optimal graph that is not super-λ_k.

A processor interconnection network or a communication network is conveniently modeled by a graph. It is well known that when studying network reliability, one often considers a network model [2] whose vertices are perfectly reliable while edges may fail independently with the same probability. The edge connectivity is an important measurement for reliability and fault tolerance of this network. In order to estimate more precisely the reliability of networks, Esfahanian and Hakimi [2] introduced the concept of restricted edge connectivity. As a generalization of restricted edge connectivity, the concept of k-restricted edge connectivity was introduced by Fàbrega and Fiol [3]. Investigations on the restricted edge connectivity of graphs were made by several authors, for example, by [4,5,8,9,11–15].

Theorem 1.1 ([5]). *Let G be a λ'-connected graph. If*

$$\max\{d(u), d(v)\} \geq \left\lfloor \frac{\nu}{2} \right\rfloor - 1$$

for each pair $u,\ v \in V(G)$ with $d(u, v) = 2$, and for each triangle T, there exists at least one vertex $v \in V(T)$ such that

$$d(v) \geq \left\lfloor \frac{\nu}{2} \right\rfloor + 1,$$

then G is λ'-optimal. If the conditions still hold after replacing $d(\cdot)$ with $d(\cdot) - 1$, then G is super-λ'.

Theorem 1.2 ([9]). *Let G be a λ'-connected graph. If*

$$d(u) + d(v) \geq 2\left\lfloor \frac{\nu}{2} \right\rfloor - 3$$

for each pair $u,\ v \in V(G)$ with $d(u, v) = 2$, and for each triangle T, there exists at least one vertex $v \in V(T)$ such that

$$d(v) \geq \left\lfloor \frac{\nu}{2} \right\rfloor + 1,$$

then G is λ'-optimal.

In Section 2, we prove a generalized degree condition for graphs to be λ_k-optimal and a generalized degree condition for graphs to be super-λ_k, respectively, which extends Theorem 1.1. Moreover, we also prove a degree condition for graphs to be λ_k-optimal, which extends Theorem 1.2.

2. Main results

Lemma 2.1 ([11]). *If G is a λ_3-connected graph, then $\lambda_3(G) \leq \xi_3(G)$.*

Lemma 2.2 ([8]). *If G is a λ_4-connected graph with order $\nu \geq 11$, then $\lambda_4(G) \leq \xi_4(G)$.*

Lemma 2.3 ([9]). *Let k be a positive integer with $k \geq 5$ and let G be a λ_k-connected graph with order $\nu > k(k-1)$. Then $\lambda_k(G) \leq \xi_k(G)$.*

Lemma 2.4 ([4]). *Let G be a λ_k-connected graph with $\lambda_k(G) \leq \xi_k(G)$. Then G is λ_k-optimal if and only if either G is not λ_{k+1}-connected or G is λ_{k+1}-connected with $\lambda_{k+1}(G) \geq \xi_k(G)$; G is super-λ_k if and only if either G is not λ_{k+1}-connected or G is λ_{k+1}-connected with $\lambda_{k+1}(G) > \xi_k(G)$.*

Theorem 2.5. *Let G be a λ_k-connected graph with order $\nu > k(k-1)$. If*

$$\max\{d(u), d(v)\} \geq \left\lfloor \frac{\nu}{2} \right\rfloor + k - 2d(u, v) + 1$$