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Global security in claw-free cubic graphs

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ABSTRACT

A *secure set* in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that for any subset $X \subseteq S$, $|N[X] \cap S| \geq |N(X) - S|$. A *global secure set* $SD \subseteq V$ is a secure set that is also a dominating set, i.e., $N[SD] = V$. In this paper we investigate global secure sets that contain exactly half of the vertices of the graph. In particular we show that every hamiltonian claw-free cubic graph has such a global secure set. Moreover, we prove that in any claw-free cubic graph there is a global secure set that contains at most 5/9 of the vertices of the graph.

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1. Introduction

In this paper we study the minimum cardinality of global secure sets in claw-free cubic graphs. First we give necessary definitions. Let $G = (V, E)$ be a graph. Throughout this paper, we consider finite and undirected graphs without loops or multiple edges. An *open neighbourhood* of a vertex v is the set $N(v) = \{x \in V : vx \in E\}$, whereas the *closed neighbourhood* of v is the set $N[x] = N(x) \cup \{x\}$. Similarly, an open (closed) neighbourhood of a set $X \subseteq V$ is the set $N(X) = \bigcup_{v \in X} N(v)$ ($N[X] = N(X) \cup X$). The *degree* of a vertex v is the number of its neighbours and is denoted by $d(v)$. *Cubic graphs* are graphs in which every vertex is of degree 3. If $d(v) = 1$, then we say that v is a *pendant vertex*. A *claw* is a connected graph with 1 vertex of degree 3 and 3 pendant vertices. A graph is *claw-free* if it does not contain a claw as an induced subgraph. We say that a graph G contains a graph H if there is a subgraph (not necessarily induced) of G isomorphic to H. Let $B \subseteq V$. Then, by $\langle B \rangle$ we denote the graph induced by the vertices of B. The *length* of the path is the number of its edges. A *hamiltonian cycle* is a cycle that passes all vertices of a graph. For all undefined concepts we refer the reader to [2].

Secure sets were introduced by Brigham et al. in [1].

Definition 1 (*[1]*). Let $G = (V, E)$ be a graph. For any $S = \{s_1, s_2, \ldots, s_k\} \subseteq V$, an attack on S is any k mutually disjoint sets $A = \{A_1, A_2, \ldots, A_k\}$ for which $A_i \subseteq N[s_i] - S$, $1 \leq i \leq k$. A defence of S is any k mutually disjoint sets $D = \{D_1, D_2, \ldots, D_k\}$ for which $D_i \subseteq N[s_i] \cap S$, $1 \leq i \leq k$. Attack A is defendable if there exists a defence D such that $|D_i| \geq |A_i|$ for $1 \leq i \leq k$. The set S is secure if and only if every attack on S is defendable.

Let $S = \{s_1, s_2, \ldots, s_k\}$ be a secure set of $G = (V, E)$. Following the authors of [1] we say that the members of S are the *defenders* and the vertices of $N[S] - S$ are the *attackers*. Let $A = \{A_1, A_2, \ldots, A_k\}$ be an attack on S and $D = \{D_1, D_2, \ldots, D_k\}$ be a defence against A. We say that a vertex $v \in N[s_i] - S$ *attacks* $s_i \in S$ if $v \in A_i$. Similarly, we say that a vertex $s_j \in S$ *defends* $s_i \in S$ if $s_j \in D_i$. An attack on S is *maximal* if every vertex of $N[S] - S$ attacks a vertex of S. Clearly, to determine whether a set of vertices is secure, it is enough to find a defence for every maximal attack. Brigham et al. also proved that

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the set S is secure if and only if for every $X \subseteq S$, $|N[X] \cap S| \geq |N[X] - S|$ [1]. A short proof of this result and an interesting generalization of the definition of secure sets was recently given in [6] by Isaak, Johnson and Petrie.

A set $D \subseteq V$ is a *dominating set* if $N[D] = V$. If a secure set is also a dominating set, then we say that it is *a global secure set*. The minimum cardinality of a dominating set and a global secure set is denoted by $\gamma(G)$ and $\gamma_s(G)$, respectively. These numbers are called a *domination number* and a *global security number*. For any connected graph G of order n, $\gamma(G) \leq \lfloor n/2 \rfloor$ and $\gamma_s(G) \geq \lceil n/2 \rceil$. The graphs having domination number equal to half their order are completely characterized. Obtaining such characterization for global secure sets seems a hard task. Probably, it is even impossible. Our supposition follows from the fact that graphs for which it was proven that their global security number equals half their order do not share any specific properties. For more details search [4,7] and [8]. In this paper we prove that each hamiltonian claw-free cubic graph of order n has the global security number equal to $n/2$.

Theorem 1. *If a graph G of order n is a hamiltonian claw-free cubic graph, then $\gamma_s(G) = n/2$.*

We believe that the global security number of non-hamiltonian claw-free cubic graphs will have the same value.

Conjecture 1. *If a graph G of order n is a claw-free cubic graph, then $\gamma_s(G) = n/2$.*

As a support of this conjecture we show that any claw-free cubic graph has a global secure set with at most 5/9 of the vertices of the graph.

Theorem 2. *If G is a claw-free cubic of order n, then $\gamma_s(G) \leq 5n/9$.*

The paper is organized as follows. In Section 2, we present some properties of secure sets and global secure sets containing half of vertices of a graph. In Section 3 we consider the structure of global secure sets that contain half of vertices of a claw-free cubic graph. Our two main results, i.e., Theorems 1 and 2 we prove in Sections 4 and 5, respectively. In Section 4, we show that any hamiltonian claw-free cubic graph of order n has a global secure set of cardinality $n/2$. Whereas in Section 5, we give an upper bound on the global security number of claw-free cubic graphs. We show that any claw-free cubic graph has a global secure set with at most 5/9 of the vertices of the graph.

Throughout this paper we use the following drawing convention. In every figure the vertices of a graph that are coloured black belong to a secure set. The white vertices are outside the secure set and the status of grey vertices is not precised and is either irrelevant or is going to be determined.

2. Preliminaries

In the forthcoming sections we use the following properties of secure sets.

Proposition 1 (*[1]*)**.** *If S_1 and S_2 are vertex disjoint secure sets in the same graph, then $S_1 \cup S_2$ is a secure set.*

Observation 1. *Let S be a secure set of G. If we add a new edge e that both of its end-vertices are either in or outside S, then S is still the secure set of graph $G + e$.*

For more results about properties of secure sets we refer the reader to [1,3,5] and [9]. Next, we present some properties of global secure sets that contain exactly half of the vertices of a graph.

Proposition 2. *Let G be a graph and S be a global secure set of G such that $|V(G)| = 2|S|$. Then,*
(i) *in any maximal attack, every vertex must participate in a defence of S,*
(ii) *for every vertex $v \in S$, $N[N(v)] \cap (V - S) \neq \emptyset$.*

Proof. (i) Since S is a dominating set, every vertex of $V - S$ has a neighbour in S. Thus, in any maximal attack there are $|V(G)|/2$ attackers. If there is a vertex of S that do not participate in a defence, then the number of the attackers exceeds the number of defenders and we have a contradiction with the security of S.

(ii) Suppose conversely that there exists a vertex $v \in S$ such that $N[N(v)] \cap (V - S) = \emptyset$. By the definition of a secure set, every vertex of S can defend either itself or one of its neighbours. Since neither v nor any of its neighbours has a neighbour in $V - S$, v does not participate in any defence, which contradicts (i). ■

Lemma 1. *Let G be a graph and S be a global secure set of G such that $|V(G)| = 2|S|$, and let $S_1, \ldots, S_k$ be vertex sets of components of $\langle S \rangle$. Then, for any $i, j \in \{1, \ldots, k\}$ there does not exist a vertex $x \notin S$ such that $x \in N(S_i) \cap N(S_j)$.*

Proof. Suppose conversely that there exists a vertex $x \notin S$ and components of $\langle S \rangle$ such that $x \in N(S_i) \cap N(S_j)$. Consider a maximal attack A on S such that x attacks a vertex $y \in S_i$. Let W denote the set of the attackers of S_j. Since A is maximal and by Proposition 2 every vertex of S_j participates in a defence of S, $|W| = |S_j|$. Now let us modify the attack A in such a way that x attacks a vertex $z \in S_j$, let us denote the obtained attack by A'. Now the vertices of S_j are attacked by $|W| + 1$ attackers. Since $|W| = |S_j|$, the attack A' cannot be repelled, which contradicts the security of S. ■

Corollary 1. *Let G be a graph, S be a global secure set of G such that $|V(G)| = 2|S|$, and let S' be a vertex set of any component of $\langle S \rangle$. Then, $|S'| = |N[S'] - S|$.*

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