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On the polyhedral structure of uniform cut polytopes

José Neto

Institut Mines-Télécom, Télécom SudParis, CNRS Samovar UMR 5157, 9 Rue Charles Fourier, 91011 Evry Cedex, France

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ABSTRACT

A uniform cut polytope is defined as the convex hull of the incidence vectors of all cuts in an undirected graph G for which the cardinalities of the shores are fixed.

In this paper, we study linear descriptions of such polytopes. Complete formulations are presented for the cases when the cardinality k of one side of the cut is equal to 1 or 2. For larger values of k, investigations with relation to the shape of these polytopes are reported. We namely determine their diameter and also provide new families of facet-defining inequalities.

1. Introduction

For terminology we shall use, the reader may consult, e.g., [19,20]. Let G denote an undirected graph with node set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Given a node set $S \subseteq V(G)$, let $\delta_G(S)$ (or $\delta(S)$ when G is clear from the context) stand for the cut that is defined by S, i.e., the set of edges in $E(G)$ with exactly one endpoint in S : $\delta_G(S) = \{e \in E(G): |e \cap S| = 1\}$. The node sets S and $V(G) \setminus S$ are called the *shores* or *sides* of the cut $\delta(S)$ and the *size* of the cut $\delta(S)$ is the cardinality of the node set of one shore: $|S|$ or $n - |S|$.

The convex hull of the incidence vectors of all cuts with a prescribed size is called a *uniform cut polytope*. Given some integer k satisfying $k \leq \lfloor \frac{n}{2} \rfloor$, let $\text{CUT}_k(G)$ denote the convex hull of the incidence vectors of cuts for which one shore has cardinality k, i.e. $\text{CUT}_k(G) = \text{conv}\{\chi^{\delta_G(S)}: S \subseteq V(G), |S| = k\}$, where, for some given subset of edges $F \subseteq E(G)$, $\chi^F \in \mathbb{R}^{|E(G)|}$ denotes the incidence vector of F (i.e. $\chi^F_e = 1$ if $e \in F$ and 0 otherwise). In the particular case when the graph G is K_n: the complete graph with order n, we shall use the notation V_n (resp. E_n) for the node set (resp. edge set) and write CUT^n_k in lieu of $\text{CUT}_k(G)$.

The motivation for the present work is to develop theoretical knowledge with respect to the polyhedral structure of cut polyhedra with potential applications to solve many difficult and challenging graph partitioning problems.

Among the latter we can namely cite the well-known maximum cut and equipartition problems. Both may be formulated by linear programs having for the feasible region the cut polytope (i.e. the convex hull of the incidence vectors of all cuts in G) and the equipartition polytope (i.e. $\text{CUT}_{\lfloor \frac{n}{2} \rfloor}(G)$), respectively.

However, given that these two problems are NP-hard in general [16], the formulation of the associated polytopes is complex (up to date no complete description of these polytopes is known for a general graph G) and has led to many polyhedral studies (e.g., [2,1,3,4,6–8,10,11,9,13–15]) aiming at determining families of facet-defining inequalities for these polytopes (and some extensions) which could then be used in order to obtain tight relaxations for these (or other related) problems.

Obviously, as uniform cut polytopes are all contained in the cut polytope, any inequality that is valid for the cut polytope is also valid for all uniform cut polytopes (and naturally, the converse does not hold in general). However, in some cases,

E-mail address: Jose.Neto@telecom-sudparis.eu.

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uniform cut polytopes may have a much more simple linear description so that it may be valuable, when solving optimization problems on uniform cut polytopes (or potentially in order to solve an optimization problem on the cut polytope by decomposing it according to different sizes of the shores), to investigate more thoroughly their precise polyhedral structure.

The polytope $\mathrm{CUT}_{\lfloor \frac{n}{2} \rfloor}$ was studied by Conforti et al. [6,7]. They introduced several classes of inequalities and showed that some are facet-defining. Their work was extended by de Souza and Laurent [11], who introduced several classes of facets arising from generalizations of an inequality from [6,7] that is based on a cycle. Also Deza et al. [12] propose a description of CUT_1^n (which we correct hereafter). Investigations have been carried out by the author in a previous paper [18] aiming at extending results by Conforti et al. [6,7] to uniform cut polytopes of the form CUT_k^n with $k < \lfloor \frac{n}{2} \rfloor$ and the purpose of the present paper is to provide further insights into the polyhedral structure of this family of polytopes.

The contribution of the present paper is 4-fold:

- present complete descriptions for small uniform cut polytopes corresponding to the cases $k = 1, 2$. These results (though independently "re-discovered" by the author) have been determined earlier by Deza, Fukuda and Laurent [12]. However the formulation they give in this reference for the uniform cut polytope CUT_1^n is incomplete. Additionally we show that facet-defining inequalities of the cut polytope CUT_2^n arise as a special case of a larger class of facet inequalities for uniform cut polytopes;
- provide some insights with respect to the shape of the family of uniform cut polytopes and determine their diameter;
- show that some families of facet-defining inequalities for the cut polytope (different from the ones studied in [18]) also define facets of uniform cut polytopes;
- introduce two families of facet-defining inequalities for uniform-cut polytopes.

This paper is organized as follows. In Section 2 some preliminary facts with respect to the polyhedral structure of uniform cut polytopes are mentioned and their diameter is determined. Then, we present different families of facet-defining inequalities in Section 3. Finally, we report minimal descriptions for the uniform cut polytopes CUT_1^n (Section 4) and CUT_2^n (Section 5).

2. On the geometry of uniform cut polytopes

In this section we report on general features of the geometry of uniform cut polytopes and determine their diameter.

2.1. Some structural aspects

The dimension of the uniform cut polytopes CUT_k^n is $\binom{n}{2} - 1$ for $2 \le k \le \frac{n-1}{2}$ (see e.g. [18]). They correspond to the intersection of the uniform cut cone $\hat{\mathrm{CUT}}_k^n = \mathrm{cone}\{\chi^{\delta_G(S)} \colon S \subseteq V(G), |S| = k\}$ (i.e. the conic hull of the incidence vectors of cuts of the form $\delta(S)$ with $S \subset V(G)$ and $|S| = k$) with the hyperplane that is defined by the following equation (cardinality constraint)

$$\sum_{e \in E_n} x_e = k(n-k). \tag{1}$$

There is a one-to-one correspondence between the facets of the uniform cut cone $\hat{\mathrm{CUT}}_k^n$ and of the uniform cut polytope CUT_k^n, and a linear formulation of $\hat{\mathrm{CUT}}_k^n$ follows from one of CUT_k^n by combining all the facet-defining inequalities of CUT_k^n with equation (1) in order to get a homogeneous system of inequalities (i.e. with zero right-hand sides only).

Notice that some classical connections between the cut cone $\hat{\mathrm{CUT}}^n$ and the cut polytope CUT^n do not hold in the uniform case. In particular, a formulation of the cut polytope CUT^n can be obtained from one of the cut cone $\hat{\mathrm{CUT}}^n$ using the so-called *switching* operation (see [13]). In fact, using this operator, the whole description of the cut polytope can be obtained from the set of all the facet-defining inequalities intersecting at some extreme point of CUT^n. However the switching operation is not valid in the uniform case. This stems from the fact that the family of the cuts with a prescribed size is not closed under symmetric difference. This may suggest that less symmetry arises in the formulations of uniform cut cones and polytopes and that the latter might have a simpler polyhedral structure w.r.t. the number of facets.

We mention in Table 1 the number of facets of some small uniform cut polytopes CUT_k^n ($k = 1, 2, 3$), cut cones $\hat{\mathrm{CUT}}^n$ and cut polytopes CUT_n. Results concerning $\hat{\mathrm{CUT}}^n$ and CUT^n are from [13], and others have been obtained using PORTA [5].

To the author's present knowledge, there is no general relation reported in the literature, between the facets of uniform cut polytopes of the form $\mathrm{CUT}_{k_1}^{n_1}$ and $\mathrm{CUT}_{k_2}^{n_2}$ with $n_1 \neq n_2$ and/or $k_1 \neq k_2$. Let us just mention the following simple case: for n even there is a 1-to-1 correspondence between the facets of $\mathrm{CUT}_{\frac{n}{2}}^n$ and $\mathrm{CUT}_{\frac{n}{2}-1}^{n-1}$. This follows from the fact that the polytope $\mathrm{CUT}_{\frac{n}{2}-1}^{n-1}$ may be interpreted as the projection of $\mathrm{CUT}_{\frac{n}{2}}^n$ onto the set of edges of the subgraph of K_n that is induced by the node set $\{v_1, \ldots, v_{n-1}\}$ and by the fact that $\mathrm{CUT}_{\frac{n}{2}}^n$ has dimension $\binom{n}{2} - n$: all its extreme points satisfy the following set of n equations: $\sum_{j \in V(K_n): j \neq i} x_{ij} = \frac{n}{2},\ \forall i \in V(K_n)$. So given any inequality that is facet-defining for $\mathrm{CUT}_{\frac{n}{2}}^n$ it is possible to