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Permutation bigraphs and interval containments

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ABSTRACT

A bipartite graph with partite sets X and Y is a *permutation bigraph* if there are two linear orderings of its vertices such that xy is an edge for $x \in X$ and $y \in Y$ if and only if x appears later than y in the first ordering and earlier than y in the second ordering. We characterize permutation bigraphs in terms of representations using intervals. We determine which permutation bigraphs are interval bigraphs or indifference bigraphs in terms of the defining linear orderings. Finally, we show that interval containment posets are precisely those whose comparability bigraphs are permutation bigraphs, via a theorem showing that a directed version of interval containment provides no more generality than ordinary interval containment representation of posets.

1. Introduction

A *permutation graph* is an undirected graph G representable by two orderings of $V(G)$ so that vertices are adjacent if and only if they appear in opposite order in the two orderings. Permutation graphs have been heavily studied ([3,10,26] summarize early work); there are fast recognition algorithms [7,16,19], and many NP-complete optimization problems for general graphs admit fast algorithms for permutation graphs ([5,17,20], etc.).

The definition has several well-known equivalent restatements. The *comparability graph* of a partially ordered set (poset) P has a vertex for each element, with x and y adjacent if $x \le y$ or $y \le x$ in P. A poset has *dimension* 2 if its elements admit two linear orderings such that $x \le y$ in P if and only if x precedes y in both orderings. To prove the equivalences in Theorem 1.1, reversing one of the orderings used yields (d)⇔(b), (d)⇒(c), and (a)⇔(b), with (b)⇒(a) because intervals can be expanded simultaneously to have a common point without changing containments. The implication (c)⇒(d) is more subtle.

Theorem 1.1 ([8,9,11]). *For a graph G, the following conditions are equivalent:*

(a) *G is a permutation graph;*
(b) *G is the containment graph of a family of intervals in $\mathbb{R}$;*
(c) *Both G and its complement are transitively orientable (that is, comparability graphs);*
(d) *G is the comparability graph of a poset of dimension at most 2.*

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The subclass of permutation graphs consisting of those that are bipartite has also been studied, with structure studied initially in [27,28], optimization problems in [1,4,12,13], and enumeration in [21]. We study a different class of bipartite graphs that properly contains the bipartite permutation graphs.

Definition 1.2. An X, Y-*bigraph* is a bipartite graph with *bipartition* into parts X and Y. A *permutation bigraph* is an X, Y-bigraph G that can be represented by two vertex orderings L and π so that $xy \in E(G)$ for $x \in X$ and $y \in Y$ if and only if x follows y in L and x precedes y in π. We call (L, π) a *permutation model* for G.

If the first ordering L is viewed as numbering the vertices from 1 through n, then (X, Y) becomes a partition of $\{1, \dots, n\}$, and the edges of G correspond to inversions in the second ordering π such that the first (larger) element of the inversion comes from X. Thus, instead of specifying (L, π), we can just specify a single ordering π on $[n]$ plus the subset of $[n]$ occupied by X, where $[n] = \{1, \dots, n\}$. The ordering L is then just the reference numbering of the vertices. Similarly, a permutation graph is specified by a single ordering π when we view the first ordering as a reference numbering.

The permutation bigraph on (X, Y) with model (L, π) is a subgraph of the permutation graph with that model, keeping as edges only some of the inversions in π. The *bipermutation bigraph* on (X, Y) with model (L, π) keeps as edges all inversions between elements of X and Y no matter which part contributes the first element; the first "bi" indicates the symmetric treatment of X and Y. This X, Y-bigraph arises by just deleting all edges within X or within Y from the permutation graph with model (L, π). It turns out that the more restricted concept of permutation bigraph is more fruitful.

Interchanging the roles of X and Y while keeping (L, π) the same changes the permutation bigraph generated by the model. The graph whose edges are the inversions where the element of Y comes first also has a permutation model with respect to the ordered partition (X, Y); just reverse both orderings. Thus the bipermutation bigraph on (X, Y) with model (L, π) is the union of the permutation bigraph with model (L, π) and the permutation bigraph generated by the model in which each of L and π is reversed.

Our initial characterizations of permutation bigraphs parallel those of permutation graphs and are quite easy. We need several other families of graphs.

Definition 1.3. An *interval containment bigraph* is an X, Y-bigraph representable by assigning each vertex an interval in $\mathbb{R}$ so that vertices $x \in X$ and $y \in Y$ are adjacent if and only if the interval for y contains the interval for x. A *chain graph* is an X, Y-bigraph in which the neighborhoods of one partite set (and hence also the other) form a chain under inclusion. A *circular-arc graph* is the intersection graph of a family of arcs on a circle. The *intersection* of two graphs G and H with $V(G) = V(H)$ is the graph $G \cap H$ with vertex set $V(G)$ given by $E(G \cap H) = E(G) \cap E(H)$.

Theorem 1.4. *For an X, Y-bigraph graph B, the following conditions are equivalent:*

(a) *B is a permutation bigraph;*
(b) *B is the intersection of two chain graphs with the same bipartition;*
(c) *B is an interval containment bigraph;*
(d) *B is bipartite and its complement is a circular-arc graph.*

In Section 2, we prove Theorem 1.4 and provide examples relating permutation bigraphs to other classes. In particular, two well-studied subclasses of permutation bigraphs are the following.

Definition 1.5. An *interval bigraph* is an X, Y-bigraph representable by assigning each vertex an interval in $\mathbb{R}$ so that vertices $x \in X$ and $y \in Y$ are adjacent if and only if their intervals intersect. An *indifference bigraph* is an interval bigraph having an interval representation in which all intervals have the same length.

An interval X, Y-bigraph arises from an interval graph with vertex set $X \cup Y$ by deleting the edges within X and within Y, just as bipermutation bigraphs arise from permutation graphs. Sen, Das, Roy, and West [22] proved that the interval bigraphs are the intersections of two chain graphs whose union is a complete bipartite graph with the same bipartition. Thus by Theorem 1.4 every interval bigraph is a permutation bigraph. Steiner [28] proved that the indifference bigraphs are just the bipartite permutation graphs. In Section 3 we characterize the permutation models for interval bigraphs and for indifference bigraphs. For a permutation σ of a set $X \cup Y$ of vertices, let σ^* denote the reverse permutation.

Theorem 1.6. *A permutation bigraph is an interval bigraph if and only if it has a permutation model (L, π) such that the permutation bigraph with model (L^*, π^*) has no edges.*

Theorem 1.7. *A permutation bigraph B is an indifference bigraph if and only if it has a permutation model (L, π) such that the model (L^*, π^*) generates no edges and each partite set appears in the same order in π as in L.*

Finally, in Section 4 we relate permutation bigraphs to posets. Our most difficult result is a direct proof of a characterization of posets whose comparability digraphs are interval containment digraphs. The result was proved originally by Sen, Sanyal, and West [24], but there the hard direction relied on a result of Bouchet [2] characterizing the dimension of posets. The proof here uses only a simpler and better known theorem of Cogis [6], plus our results from Section 2. For the statement, we need other structures associated with posets.