Discrete Applied Mathematics 175 (2014) 79–86

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

[1, 2]-domination in graphs ☆

Xiaojing Yang, Baoyindureng Wu *

College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, PR China

ARTICLE INFO

Article history:
Received 8 October 2013
Received in revised form 19 May 2014
Accepted 21 May 2014
Available online 6 June 2014

Keywords:
[1, 2]-sets
Domination
Lexicographic product
Nordhaus–Gaddum
Restrained domination

ABSTRACT

A vertex subset D of a graph $G = (V, E)$ is a [1, 2]-set if, $1 \leq |N(v) \cap D| \leq 2$ for every vertex $v \in V \setminus D$, that is, each vertex $v \in V \setminus D$ is adjacent to either one or two vertices in D. The minimum cardinality of a [1, 2]-set of G, denoted by $\gamma_{[1,2]}(G)$, is called the [1, 2]-domination number of G. Chellali et al. (2013) showed that there exist graphs G of order n with $\gamma_{[1,2]}(G) = n$. But, the complete characterization of such graphs seems to be a difficult task. As responding to some open questions posed by Chellali et al., we further show that such graphs exist even among some special families of graphs, such as planar graphs, bipartite graphs (triangle-free graphs). It is also shown that for a tree T of order $n \geq 3$ with k leaves, if $d_T(v) \geq 4$ for any non-leaf vertex v, then $\gamma_{[1,2]}(T) = n - k$. In addition, Nordhaus–Gaddum-type inequalities are established for the [1, 2]-domination numbers of graphs. Thereby, we solve several open problems posed by Chellali et al.

1. Introduction

We consider undirected finite simple graphs only, and refer to [10] for undefined notations and terminology. Let $G = (V, E)$ be a graph. The order of G is $|V|$. The neighborhood $N(v)$ of a vertex $v \in V(G)$ is $\{u : uv \in E(G)\}$. The degree $d_G(v)$ (or simply $d(v)$) of v is the number of edges incident with v in G. Since G is simple, $d(v) = |N(v)|$. The open neighborhood of a set $S \subseteq V$ is $N(S) = \cup_{v \in S} N(v)$, and the subgraph of G induced by the vertices in S is denoted by $G[S]$. A vertex is called a leaf in a tree if it has degree 1. The minimum and maximum degrees of a vertex in a graph G are denoted $\delta(G)$ and $\Delta(G)$, respectively.

The complement of G, denoted by $\overline{G}$, is the graph with vertex set $V(G)$ and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

A set $D \subseteq V$ is a dominating set of a graph G if for every $v \in V$, either $v \in D$ or $v \in N(u)$ for some vertex $u \in D$. The minimum cardinality of a dominating set in a graph G is called the domination number of G, and is denoted by $\gamma(G)$. We refer to the two excellent books [8,7] for consulting results and problems on the domination of graphs.

For any two integers j and k, a subset $D \subseteq V$ in a graph $G = (V, E)$ is a $[j, k]$-set if, for every vertex $v \in V \setminus D, j \leq |N(v) \cap D| \leq k$. That is, each vertex $v \in V \setminus D$ is adjacent to at least j vertices, but not more than k vertices in D. For $j \geq 1$, a $[j, k]$-set D is a dominating set, since every vertex in $V \setminus D$ has at least one neighbor in D. The notions of $[j, k]$-set and $[j, k]$-domination were recently introduced by Chellali et al. [3]. Its special case when $j = 1$ and $k = 2$ was first investigated by Dejter [4]. For more general concepts, called set-restricted dominating set and set-restricted domination number, we refer to Amin and Slater [1,2].

☆ Research was supported by NSFC (No. 11161046) and by the Scientific and Technological Innovation Young Personnel Training Project of Xinjiang (No. 2013721012).

* Corresponding author.
E-mail addresses: baoyin@xju.edu.cn, baoyinwu@gmail.com, wubaoyin@hotmail.com (B. Wu).

http://dx.doi.org/10.1016/j.dam.2014.05.035

Definition. For $k \geq 1$, the $[1, k]$-domination number in a graph G, denoted by $\gamma_{[1,k]}(G)$, equals the minimum cardinality of a $[1, k]$-set in G. A $[1, k]$-set with cardinality $\gamma_{[1,k]}(G)$ is called a $\gamma_{[1,k]}$-set of G.

Note that every graph G has a $[1, 2]$-set, since the set $D = V$ vacuously satisfies the condition that $1 \leq |N(v) \cap D| \leq 2$ for every vertex $v \in V \setminus D$. In [3], Chellali et al. studied $[1, 2]$-sets in graphs and posed a number of open problems. Some of them are solved in this paper, as listed below.

Question 1. Is it true for planar graphs G that $\gamma_{[1,2]}(G) < n$?

Question 2. Is it true for bipartite graphs G that $\gamma_{[1,2]}(G) < n$?

Question 3. Is it true for triangle-free graphs G that $\gamma_{[1,2]}(G) < n$?

Question 4. If T is a tree of order n with k leaves, then $\gamma_{[1,2]}(T) \leq n - k$. For which trees is this bound sharp?

Question 5. Can you bound $\gamma_{[1,2]}(G) + \gamma_{[1,2]}(\overline{G})$?

Indeed, we answer the first three questions in the negative, give a sufficient condition for a tree satisfying the equality in Question 4, and establish the Nordhaus–Gaddum-type inequalities by giving sharp upper and lower bounds for $\gamma_{[1,2]}(G) + \gamma_{[1,2]}(\overline{G})$ and $\gamma_{[1,2]}(G)\gamma_{[1,2]}(\overline{G})$.

2. Graphs G of order n with $\gamma_{[1,2]}(G) = n$

We begin with two basic facts regarding $[1, 2]$-domination numbers of graphs.

Lemma 2.1 ([3]). *For any graph G of order n, $\gamma(G) \leq \gamma_{[1,2]}(G) \leq n$.*

Lemma 2.2 ([3]). *For any connected graph G of order $n \geq 2$, if $\delta(G) \leq 2$, then $\gamma_{[1,2]}(G) < n$.*

It is well-known [9] that for a connected graph G of order n, if $\delta(G) \geq 1$, then $\gamma(G) \leq \frac{n}{2}$. The situation for $\gamma_{[1,2]}(G)$ is different. The authors in [3] showed that there exist graphs G with order n and $\delta(G) \geq 1$ for which $\gamma_{[1,2]}(G) = n$. In this section, we will find more graphs G of order n with $\gamma_{[1,2]}(G) = n$, which will be used in establishing Nordhaus–Gaddum-type inequalities for $[1, 2]$-domination number in Section 3.

Let p and k be two integers with $p \geq k + 2$. Let $G_{p,k}$ be the graph obtained from a complete graph K_p as follows. For every k-element subset S of the vertices $V(K_p)$, we add a new vertex x_S and the edges $x_S u$ for all $u \in S$. Note that the total number of added vertices is $\binom{p}{k}$, and the number of added edges is $k\binom{p}{k}$. We remark that the result of the following theorem for $k = 3$ is due to Chellali et al. [3], and the proofs are basically same.

Theorem 2.3. *Let p and k be two integers, and $G_{p,k}$ be the graph of order n in the above construction. If $p \geq k + 2$ and $k \geq 3$, then $\gamma_{[1,2]}(G_{p,k}) = |V(G_{p,k})| = n$.*

Proof. In the construction of $G_{p,k}$, let $P = V(K_p) = \{u_1, u_2, \ldots, u_p\}$ and X be the set of $\binom{p}{k}$ added vertices, that is, $X = \{x_S : S$ is a k-element subset of $V(K_p)\}$. Let D be a $\gamma_{[1,2]}$-set of $G_{p,k}$, and assume that $\gamma_{[1,2]}(G_{p,k}) = |D| < n$.

Assume that $P \subseteq D$. By our assumption that $\gamma_{[1,2]}(G_{p,k}) < n$, there is a vertex $x_S \notin D$. But then x_S is dominated by k vertices of $D \cap P$, a contradiction. Hence, we may assume that at least one vertex u_i in $P \setminus D$. If $|D \cap P| \geq 3$, then, since $p \geq 5$, $u_i \in P \setminus D$ is dominated at least three times by D, a contradiction. Thus, $|D \cap P| \leq 2$.

We first assume that no vertex of P is in D. Then $X \subseteq D$, because every vertex x_S is in D to be dominated by itself. Since, for $1 \leq i \leq p$, $u_i \in P \setminus D$ and u_i is in $\binom{p-1}{k-1} \geq 6$ subsets of P of cardinality k, it follows that u_i has $\binom{p-1}{k-1} \geq 6$ neighbors in D, a contradiction.

Next assume that $|D \cap P| = 1$, and let $u_1 \in D$. Thus, D contains every vertex x_S for all k-element subsets S of P such that $u_1 \notin S$. Then $u_2 \in P \setminus D$, and the cardinality of the k-element subsets which contain u_2 but do not contain u_1 is $\binom{p-2}{k-1} \geq k \geq 3$. It follows that vertex u_2 is dominated by at least three vertices in D.

Finally, suppose that $|D \cap P| = 2$, and let $u_1, u_2 \in D$. Thus, D contains every vertex x_S for all k-element subsets S of P such that $u_1 \notin S$ and $u_2 \notin S$. Then $u_3 \in P \setminus D$, and the number of the k-element subsets which contain u_3 but do not contain u_1 or u_2 is $\binom{p-3}{k-1} \geq 1$. It follows that vertex u_3 is dominated by at least three vertices in D, a contradiction.

Therefore, we conclude that $\gamma_{[1,2]}(G_{p,k}) = |V(G_{p,k})| = n$. $\square$

For two simple graphs G and H, their lexicographic product $G \circ H$ is the simple graph with vertex set $V(G) \times V(H)$, in which (u, v) and (u', v') are adjacent if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. This amounts to replacing each vertex v of G by a copy of H, called H_v, and adding all possible edges between the vertices of H_v and H_w if and only if $vw \in E(G)$.

The following result is interesting for lexicographic product of graphs.