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Note

A note on 1-planar graphs

Eyal Ackerman

Department of Mathematics, Physics, and Computer Science, University of Haifa at Oranim, Tivon 36006, Israel

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ABSTRACT

A graph is 1-*planar* if it can be drawn in the plane such that each of its edges is crossed at most once. We prove a conjecture of Czap and Hudák (2013) stating that the edge set of every 1-planar graph can be decomposed into a planar graph and a forest. We also provide simple proofs for the following recent results: (i) an n-vertex graph that admits a 1-planar drawing with straight-line edges has at most $4n - 9$ edges (Didimo, 2013); and (ii) every drawing of a *maximally dense right angle crossing* graph is 1-planar (Eades and Liotta, 2013).

1. Introduction

In a *drawing* of a graph in the plane its vertices are represented as distinct points and its edges as Jordan arcs that connect corresponding points and do not contain any other vertex as an interior point. Any two edges in a drawing of a graph have a finite number of intersection points. Every intersection point of two edges is either a vertex that is common to both edges, or a crossing point at which one edge passes from one side of the other edge to its other side. A drawing of a graph is 1-*planar* if each of the edges is crossed at most once. If a graph has a 1-planar drawing, then it is 1-*planar*.

The notion of 1-planarity was introduced in 1965 by Ringel [19], and since then many properties of 1-planar graphs have been studied (see, e.g., [1–4,6–8,12,14,15,17]). It is known that the maximum number of edges in an n-vertex 1-planar graph is $4n - 8$ [14,18,20], and that this bound is tight, that is, for any $n \geq 12$ there exists an n-vertex 1-planar graph with $4n - 8$ edges [18].

Czap and Hudák [7] showed that if an n-vertex 1-planar graph has the maximum number of edges, namely $4n - 8$, then its edge set can be decomposed into two subsets, such that one of them induces a planar graph and the other induces a forest. We prove their conjecture that this holds for every 1-planar graph.

Theorem 1. *Let $G = (V, E)$ be a 1-planar graph. Then there is a partition of E into two subsets A and B such that A induces a planar graph and B induces a forest.*

Note that it is not always possible to add an edge to a 1-planar graph with less than $4n - 8$ edges. Indeed, Brandenburg et al. [4] showed that there are 1-planar graphs with n vertices and only $\frac{45}{17}n + O(1)$ edges, such that adding an edge to any such graph results in a graph that is no longer 1-planar. Therefore, one cannot conclude Theorem 1 simply because it holds for 1-planar graphs with the maximum possible number of edges.

The converse of Theorem 1 is false: a graph G that can be partitioned into a planar graph and a forest is not necessarily 1-planar. In fact, even when the forest consists of a single edge, G might not be 1-planar and deciding its 1-planarity is NP-complete [5].

Apart from Theorem 1, this note contains simple proofs of two recent results related to 1-planar graphs. The first result is due to Didimo [8]:

E-mail address: ackerman@sci.haifa.ac.il.

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Theorem 2 (*[8]*)**.** *A graph on $n \geq 3$ vertices that can be drawn in the plane with straight-line edges such that every edge is crossed at most once has at most $4n - 9$ edges.*

The second result, due to Eades and Liotta [13], concerns drawings of graphs with right angle crossings. A *right angle crossing* (RAC) drawing of a graph is a drawing with straight-line edges that may cross each other only at a right angle. A RAC graph is a graph that admits a RAC drawing. The class of RAC graphs was introduced by Didimo et al. [9], following experiments showing that large angle crossings are visually appealing [16]. They proved that an n-vertex RAC graph has at most $4n - 10$ edges and that this bound is tight, namely, there are RAC graphs with that many edges. We say that such RAC graphs are *maximally dense*.

Eades and Liotta [13] recently showed that every RAC drawing of a maximally dense RAC graph must be 1-planar. They also showed that there are RAC graphs that do not admit 1-planar RAC drawings, and that there are graphs with $4n - 10$ edges that admit 1-planar drawings but no RAC drawing. For further results on RAC graphs and related problems see a recent survey of Didimo and Liotta [10]. Here we provide a different and shorter proof of the main theorem in [13]. Moreover, our result is a bit more general.

Theorem 3. *Let D be a RAC drawing of an n-vertex graph $G = (V, E)$, $n \geq 3$, such that there is an edge $e \in E$ which is crossed $k \geq 1$ times in D. Then $|E| \leq 4n - 9 - k$.*

Corollary 4 (*[13]*)**.** *If G is a maximally dense RAC graph, then every RAC drawing of G is 1-planar.*

Organization. Theorem 1 is proved in Section 2, while Theorems 2 and 3 are proved in Section 3.

2. Proof of Theorem 1

Let $G = (V, E)$ be a 1-planar graph drawn in the plane such that no edge is crossed more than once. Henceforth we do not distinguish G from its drawing. We may assume without loss of generality that if two edges cross in G, then they do not share a common vertex. Otherwise, these edges can be redrawn such that this crossing is eliminated and no new crossing is introduced (and the abstract graph remains the same).

Therefore, every crossing involves two edges with four distinct endpoints. In such a case we show that E can be partitioned into two subsets A and B, such that A induces a plane graph and B induces a plane forest. Note that this proves a slightly stronger statement than the one stated in Theorem 1.

Let p be a crossing point and let u and v be two vertices of the edges that cross at p, such that (u, v) is not one of these edges. Then we can draw a new edge (u, v) without introducing any crossings by following the two edges that cross at p from u and v until they meet in a close neighborhood of p. For every crossing point p and every such u and v, we draw a new edge (u, v) as described. Note that the new drawing might contain parallel edges. Denote the new (multi)graph by $G' = (V, E')$ and let $E'_i \subseteq E'$ be the edges in E' that are crossed exactly i times, for $i = 0, 1$.

Call a face whose boundary is a simple cycle of length four a *quadrangle*. A *chord* is a new edge that is drawn within a face and connects two of its vertices that are not consecutive on the boundary of the face. The proof of Theorem 1 follows from the next claim.

Proposition 2.1. *For every plane multigraph $H = (V, E)$ and every pair of adjacent vertices $x, y \in V$ it is possible to add a chord to every quadrangle in H, such that the graph induced by the chords is a forest in which there is no path between x and y.*

Indeed, denote by G'_0 the plane multigraph induced by E'_0. Then every pair of crossing edges in G is the possible chords of a distinct quadrangle in G'_0. Applying Proposition 2.1 to G'_0 we obtain a set $B \subset E'_1$ of edges that induce a plane forest. Setting $A = (E'_0 \cap E) \cup (E'_1 \setminus B)$ we obtain the subsets A and B as required. It remains to prove Proposition 2.1.

Proof of Proposition 2.1. We may assume without loss of generality that H does not contain faces of size greater than four, for such faces can be triangulated without changing the set of quadrangles in H. We prove the claim by induction on the number of quadrangles in H. If there are no quadrangles, then the claim trivially holds. Otherwise, let f be a quadrangle and let v_0, v_1, v_2, v_3 be the vertices on the boundary of f, listed in their clockwise order around f. We consider several cases.

Case 1: There is no other face but f that is incident to both v_0 and v_2 (see Fig. 1(a)). Note that this implies that v_0 and v_2 are not adjacent. We add the edge (v_0, v_2) within f and immediately contract it. Denote by H' the resulting graph, and let v be the vertex into which v_0 and v_2 are merged (see Fig. 1(b)). Comparing H and H' we observe that f was replaced by two faces of size two, and the size of every other face and the number of vertices on its boundary have not changed. We now apply the induction hypothesis on H' with the same pair of adjacent vertices x, y (if one of v_0 or v_2 is in $\{x, y\}$, then v 'plays' their role in H'). Then, we 'uncontract' the edge (v_0, v_2) and add it to the set of chords (see Fig. 1(c)). Since every quadrangle in H except f is also a quadrangle in H', there is indeed a chord now in every quadrangle. Note also that if the graph induced by the chords of H contains a cycle or an x–y path, then so does the graph induced by the chords of H'.

Note that if there was another face but f that is incident to both v_0 and v_2, then merging them would change the number of (distinct) vertices on the boundary of that face. Specifically, we could destroy two quadrangles instead of just one. Therefore, such a case should be handled with care.