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Note

A characterization of substar graphs

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ABSTRACT

The intersection graphs of stars in some tree are known as substar graphs. In this paper we give a characterization of substar graphs by the list of minimal forbidden induced subgraphs. This corrects a flaw in the main result of Chang, Jacobson, Monma and West (1993) and this leads to a different list of minimal forbidden induced subgraphs.

1. Introduction

A graph G is *chordal*, if it has no induced cycle of order at least four. Chordal graphs are one of the most fundamental graph classes and well investigated [1]. For a family $\mathcal{M}$ of sets, an *$\mathcal{M}$-intersection representation* of a graph G is a function $f : V(G) \to \mathcal{M}$ such that two distinct vertices x and y are adjacent if and only if $f(x) \cap f(y) \neq \emptyset$. If G has an $\mathcal{M}$-intersection representation, then G is an *$\mathcal{M}$-graph*.

It is a classic result [5] in graph theory that every chordal graph is a $\mathcal{T}$-graph, where $\mathcal{T}$ is the set of all subtrees of some tree. There are several important and well investigated subclasses of chordal graphs. Let $\mathcal{T}_P$ be the set of all paths in some tree. We call a $\mathcal{T}_P$-graph a *path graph*. Path graphs are investigated in many different varieties [6,7]. If $\mathcal{T}_I$ is the set of all subpaths of some path, then the class of $\mathcal{T}_I$-graphs is known as *interval graphs*, which are another very important graph class [4].

A *star* is a tree such that there is a vertex that is adjacent to all other vertices of the tree. Let $\mathcal{T}_S$ be the set of all substars in some tree T. We call a $\mathcal{T}_S$-graph a *substar graph*. Chang et al. [3] claimed to characterize substar graphs by a finite list of forbidden induced subgraphs. Since there are infinitely many minimal forbidden induced subgraphs for substar graphs, their claim is false. In this paper we give a characterization of substar graphs by the infinite list of minimal forbidden induced subgraphs.

Cerioli and Szwarcfiter [2] characterized a subclass of substar graphs, which is the class of starlike graphs. These are intersection graphs of substars of a star.

We start by introducing some basic notation. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. A subset Q of the vertices of G is a *clique* in G if every pair of vertices of Q is adjacent, and Q is a *maximal clique* in G if it is a clique not properly contained in another clique of G. Let the number of edges in a longest shortest path in G be the *diameter* of G. For $k \in \mathbb{N}$, we write $[k]$ for the set $\{1, \ldots, k\}$.

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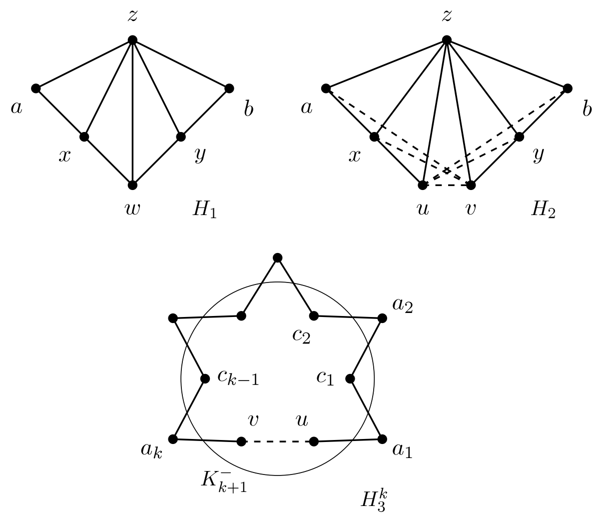

Fig. 1. The set $\mathcal{S}$ of minimal forbidden induced subgraphs for substar graphs.

2. Results

Before we state and prove our main result, we first introduce the set $\mathcal{S}$ of minimal forbidden induced subgraphs for substar graphs. See Fig. 1 for an illustration. The graph H_1 needs no further explanation. The graphs of type H_2 are as in Fig. 1 and the dashed edges indicate the following three possibilities:

- $N_{H_2}(u) \cap \{v, y, b\} = N_{H_2}(v) \cap \{u, x, a\} = \emptyset$,
- $N_{H_2}(u) \cap \{v, y, b\} = \{v\}$ and $N_{H_2}(v) \cap \{u, x, a\} = \{u, x, a\}$, or
- $N_{H_2}(u) \cap \{v, y, b\} = \{v, y, b\}$ and $N_{H_2}(v) \cap \{u, x, a\} = \{u, x, a\}$.

The graph H_3^k for some $k \geq 3$ satisfies $V(H_3^k) = \{u, v, c_1, \ldots, c_{k-1}, a_1, \ldots, a_k\}$ and

$$E(H_3^k) = \left(\left(\binom{\{u, v, c_1, \ldots, c_{k-1}\}}{2}\right) \cup \{ua_1, va_k\} \cup \bigcup_{i=1}^{k-1}\{a_ic_i, c_ia_{i+1}\}\right) \setminus \{uv\}.$$

Note that the graphs H_3^k for $k \in \{3, \ldots, 6\}$ equal the forbidden subgraphs H_3, H_4, H_5, H_6, H_7 introduced by Chang et al., and moreover, the graphs H_3 and H_6 are isomorphic. The graphs H_3^k, for k at least 7, are minimal forbidden induced subgraphs for substar graphs and do not contain any graph as an induced subgraph from the list of Chang et al.

As already mentioned above, a chordal graph G has a $\mathcal{T}$-representation, that is, there is a tree T and a function $f : V(G) \to \mathcal{T}$, where $\mathcal{T}$ is the set of all the subtrees of T and two distinct vertices v, w are adjacent if and only if $f(v) \cap f(w) \neq \emptyset$. We say the tuple (T, f) is a *tree representation* of G. It is well known and not difficult to see that one can modify every tree representation (T, f) of G by edge contractions such that $f(V(G)) = T$ without increasing the diameter of any subtree of T. Therefore, from now on we may assume that a tree representation (T, f) of a graph G has the property $f(V(G)) = T$.

Let (T, f) be a tree representation of a chordal graph G and $t \in V(T)$. Obviously, $f^{-1}(t)$ is a clique Q in G. Suppose that Q is not a maximal clique of G and let Q' be a maximal clique of G that contains Q. By the Helly property of subtrees of a tree (see for example [1]) and the maximality of Q', there is a vertex $t' \in V(T)$ such that $f^{-1}(t') = V(Q')$. Let $P : tt_1 \ldots t_kt'$ be the t, t'-path in T. Note that $V(Q) \subseteq f^{-1}(t_i)$ for every $i \in [k]$. Contracting the edge tt_1 to a vertex v_{tt_1} leads to a tree T'. Define f' by $f'^{-1}(s) = f^{-1}(s)$ if $s \in V(T) \setminus \{t, t_1\}$ and let $f'^{-1}(v_{tt_1}) = f^{-1}(t_1)$. Trivially (T', f') is a tree representation of G. Repeating this process leads to a tree representation $(\tilde{T}, \tilde{f})$ of G such that for every vertex $\tilde{t} \in V(\tilde{T})$ the vertex set $\tilde{f}^{-1}(\tilde{t})$ is a maximal clique in G. If there are two vertices t and t' of $\tilde{T}$ such that $\tilde{f}^{-1}(t) = \tilde{f}^{-1}(t')$, then contracting all edges on the t, t'-path in $\tilde{T}$ and repeating this process as often as possible leads to a tree representation $(\hat{T}, \hat{f})$ of G such that there is a bijection between the maximal cliques of G and the vertices of $\hat{T}$. Note that this process does not increase the diameter of a subtree in the tree representation induced by a vertex of G. Therefore, from now on we may assume that a tree representation (T, f) of a graph G has the property $f(V(G)) = T$ and there is a bijection between the maximal cliques of G and the vertices of T.

After these preliminaries, we proceed to our results.

Lemma 1. *The graphs in $\mathcal{S}$ are not substar graphs.*