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Note

On r-equitable chromatic threshold of Kronecker products of complete graphs[☆]

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ABSTRACT

Let r and k be positive integers. A graph G is r-equitably k-colorable if its vertex set can be partitioned into k independent sets, any two of which differ in size by at most r. The r-equitable chromatic threshold of a graph G, denoted by $\chi^*_{r=}(G)$, is the minimum k such that G is r-equitably k'-colorable for all $k' \geq k$. Let $G \times H$ denote the Kronecker product of graphs G and H. In this paper, we completely determine the exact value of $\chi^*_{r=}(K_m \times K_n)$ for general m, n and r. As a consequence, we show that for $r \geq 2$, if $n \geq \frac{1}{r-1}(m+r)(m+2r-1)$ then $K_m \times K_n$ and its spanning supergraph $K_{m(n)}$ have the same r-equitable colorability, and in particular $\chi^*_{r=}(K_m \times K_n) = \chi^*_{r=}(K_{m(n)})$, where $K_{m(n)}$ is the complete m-partite graph with n vertices in each part.

1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a positive integer k, let $[k] = \{1, 2, \ldots, k\}$. A (proper) *$k$-coloring* of G is a mapping $f : V(G) \to [k]$ such that $f(x) \neq f(y)$ whenever $xy \in E(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G admits a k-coloring. We call the set $f^{-1}(i) = \{x \in V(G): f(x) = i\}$ a *color class* for each $i \in [k]$. Notice that each color class in a proper coloring is an independent set, i.e., a subset of $V(G)$ of pairwise non-adjacent vertices, and hence a k-coloring is a partition of $V(G)$ into k independent sets. For a fixed positive integer r, an *r-equitable k-coloring* of G is a k-coloring for which any two color classes differ in size by at most r. A graph is *r-equitably k-colorable* if it has an r-equitable k-coloring. The *r-equitable chromatic number* of G, denoted by $\chi_{r=}(G)$, is the smallest integer k such that G is r-equitably k-colorable. For a graph G, the *r-equitable chromatic threshold* of G, denoted by $\chi^*_{r=}(G)$, is the smallest integer k such that G is r-equitably k'-colorable for all $k' \geq k$. Although the concept of r-equitable colorability seems a natural generalization of usual equitable colorability (corresponding to $r = 1$) introduced by Meyer [9] in 1973, it was first proposed recently by Hertz and Ries [6,7], where the authors generalized the characterizations of usual equitable colorability of trees [2] and forests [1] to r-equitable colorability. Quite recently, Yen [12] proposed a necessary and sufficient condition for a complete multipartite graph G to have an r-equitable k-coloring and also gave exact values of $\chi_{r=}(G)$ and $\chi^*_{r=}(G)$. In particular, they obtained the following results for $K_{m(n)}$, where $K_{m(n)}$ denotes the complete m-partite graph with n vertices in each part.

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Lemma 1 ([12]). *For integers $n, r \geq 1$ and $k \geq m \geq 2$, $K_{m(n)}$ is r-equitably k-colorable if and only if* $\left\lceil \frac{n}{\lfloor k/m \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil k/m \rceil} \right\rfloor \leq r$.

Lemma 2 ([12]). *For integers $n, r \geq 1$ and $m \geq 2$, we have $\chi^*_{r=}(K_{m(n)}) = m\lceil \frac{n}{\theta+r} \rceil$, where θ is the minimum positive integer such that $\lfloor \frac{n}{\theta+1} \rfloor < \lceil \frac{n}{\theta+r} \rceil$.*

The special case of Lemmas 1 and 2 for $r = 1$ was obtained by Lin and Chang [8].

For two graphs G and H, the *Kronecker product* $G \times H$ of G and H is the graph with vertex set $\{(x, y): x \in V(G), y \in V(H)\}$ and edge set $\{(x, y)(x', y'): xx' \in E(G)$ and $yy' \in E(H)\}$. In this paper, we analyze the r-equitable colorability of Kronecker product of two complete graphs. We refer to [3,5,8,11] for more studies on the usual equitable colorability of Kronecker products of graphs.

In [4], Duffus et al. showed that if $m \leq n$ then $\chi(K_m \times K_n) = m$. From this result, Chen [3] got that $\chi_=(K_m \times K_n) = m$ for $m \leq n$. Indeed, let $V(K_m \times K_n) = \{(x_i, y_j): i \in [m], j \in [n]\}$. Then we can partition $V(K_m \times K_n)$ into m sets $\{(x_i, y_j): j \in [n]\}$ with $i = 1, 2, \ldots, m$, all of which have equal size and are clearly independent. Similarly, for any $r \geq 1$, $\chi_{r=}(K_m \times K_n) = m$ for $m \leq n$. However, it is much more difficult to determine the exact value of $\chi^*_{r=}(K_m \times K_n)$, even for $r = 1$.

Lemma 3 ([8]). *For positive integers $m \leq n$, we have $\chi^*_=(K_m \times K_n) \leq \lceil \frac{mn}{m+1} \rceil$.*

In the same paper, Lin and Chang determined the exact values of $\chi^*_=(K_2 \times K_n)$ and $\chi^*_=(K_3 \times K_n)$. Note that the case when $m = 1$ is trivial since $K_1 \times K_n$ is the empty graph I_n and hence $\chi^*_=(K_1 \times K_n) = 1$. Recently, those results have been improved to the following.

Theorem 4 ([10]). *For integers $n \geq m \geq 2$,*

$$\chi^*_=(K_m \times K_n) = \begin{cases} \left\lceil \dfrac{mn}{m+1} \right\rceil, & \text{if } n \equiv 2, \ldots, m-1 (\text{mod } m+1); \\ m\left\lceil \dfrac{n}{s^*} \right\rceil, & \text{if } n \equiv 0, 1, m (\text{mod } m+1), \end{cases}$$

where s^* is the minimum positive integer such that $s^* \nmid n$ and $m\lceil \frac{n}{s^*} \rceil \leq \lceil \frac{mn}{m+1} \rceil$.

From the definition of s^*, we see that $s^* \neq 1$ and hence $s^* \geq 2$. Let $\theta = s^* - 1$. Then we can restate Theorem 4 as follows.

Theorem 5. *For integers $n \geq m \geq 2$,*

$$\chi^*_=(K_m \times K_n) = \begin{cases} \left\lceil \dfrac{mn}{m+1} \right\rceil, & \text{if } n \equiv 2, \ldots, m-1 (\text{mod } m+1); \\ m\left\lceil \dfrac{n}{\theta+1} \right\rceil, & \text{if } n \equiv 0, 1, m (\text{mod } m+1), \end{cases}$$

where θ is the minimum positive integer such that $\theta + 1 \nmid n$ and $m\lceil \frac{n}{\theta+1} \rceil \leq \lceil \frac{mn}{m+1} \rceil$.

A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H is a *spanning subgraph* of G if it has the same vertex set as G.

Corollary 6. *If $n \geq m$ and $n \equiv 2, \ldots, m-1 (\text{mod } m+1)$ then $\chi^*_=(K_m \times K_n) < \chi^*_=(K_{m(n)})$.*

Proof. Since $K_m \times K_n$ is a spanning subgraph of $K_{m(n)}$, $\chi^*_=(K_m \times K_n) \leq \chi^*_=(K_{m(n)})$. Therefore, the corollary follows if we can show $\chi^*_=(K_m \times K_n) \neq \chi^*_=(K_{m(n)})$. Let $n = (m+1)s + t$ with $s = \lfloor \frac{n}{m+1} \rfloor$ and $2 \leq t \leq m-1$. We have $\lceil \frac{mn}{m+1} \rceil = \lceil \frac{m(m+1)s+mt}{m+1} \rceil = \lceil \frac{m(m+1)s+(m+1)t-t}{m+1} \rceil = ms + t + \lceil \frac{-t}{m+1} \rceil = ms + t$. By Theorem 5, $\chi^*_=(K_m \times K_n) = \lceil \frac{mn}{m+1} \rceil = ms + t$ and hence m is not a factor of $\chi^*_=(K_m \times K_n)$. On the other hand, by Lemma 2, m is a factor of $\chi^*_=(K_{m(n)})$. Therefore, $\chi^*_=(K_m \times K_n) \neq \chi^*_=(K_{m(n)})$ and hence the proof is complete. □

The main purpose of this paper is to obtain the exact value of $\chi^*_{r=}(K_m \times K_n)$ for any $r \geq 1$, which we state as the following theorem.

Theorem 7. *For any integers $n \geq m \geq 2$ and $r \geq 1$,*

$$\chi^*_{r=}(K_m \times K_n) = \begin{cases} n - r\left\lfloor \dfrac{n}{m+r} \right\rfloor, & \text{if } n \equiv 2, \ldots, m-1 (\text{mod } m+r) \text{ and} \\ & \left\lceil \dfrac{n}{\lfloor n/(m+r) \rfloor} \right\rceil - \left\lfloor \dfrac{n}{\lceil n/(m+r) \rceil} \right\rfloor > r; \\ m\left\lceil \dfrac{n}{\theta+r} \right\rceil, & \text{otherwise}, \end{cases}$$

where θ is the minimum positive integer such that $\lfloor \frac{n}{\theta+1} \rfloor < \lceil \frac{n}{\theta+r} \rceil$ and $m\lceil \frac{n}{\theta+r} \rceil \leq \min\{n - r\lfloor \frac{n}{m+r} \rfloor, m\lceil \frac{n}{m+r} \rceil\}$.