Discrete Applied Mathematics 174 (2014) 57–65

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Finding clubs in graph classes ☆

Petr A. Golovach [a,*], Pinar Heggernes [a], Dieter Kratsch [b], Arash Rafiey [c]

[a] Department of Informatics, University of Bergen, P.O. Box 7803, 5020 Bergen, Norway

[b] LITA, Université de Lorraine, 57045 Metz, France

[c] School of Computing, Simon Fraser University, Burnaby, BC, V5A 1S6, Canada

ARTICLE INFO

Article history:
Received 4 April 2013
Received in revised form 7 April 2014
Accepted 17 April 2014
Available online 29 April 2014

Keywords:
Clique
s-club
k-chordal graphs
AT-free graphs

ABSTRACT

For a positive integer s, an s-club in a graph G is a set of vertices that induces a subgraph of G of diameter at most s. We study a relation of clubs and cliques. For a positive integer s, we say that a graph class $\mathcal{G}$ has the s-clique-power property if for every graph $G \in \mathcal{G}$, every maximal clique in G^s is an s-club in G. Our main combinatorial results show that both 4-chordal graphs and AT-free graphs have the s-clique-power property for all $s \geq 2$. This has various algorithmic consequences. In particular we show that a maximum s-club in G can be computed in polynomial time when G is a chordal bipartite or a strongly chordal or a distance hereditary graph. On weakly chordal graphs, we obtain a polynomial-time algorithm when s is an odd integer, which is best possible as the problem is NP-hard for even values of s. We complement these results by proving the NP-hardness of the problem for every fixed s on 4-chordal graphs. Finally, if G is an AT-free graph, we prove that the problem can be solved in polynomial time when $s \geq 2$, which gives an interesting contrast to the fact that the problem is NP-hard for $s = 1$ on this graph class.

1. Introduction

Max Clique is one of the most fundamental problems in graph algorithms. Cliques model highly connected or correlated parts of networks and data sets, and consequently they find applications in numerous diverse fields. For many real problems, however, cliques present a too restrictive measure of connectivity (see e.g., [1,15,26,32]), and the notion of clubs were proposed to give more realistic models [3,27]. Given a graph $G = (V, E)$ on n vertices and an integer s between 1 and n, a vertex subset $S \subseteq V$ is an s-club if the subgraph of G induced by S has diameter at most s. Hence 1-clubs are exactly cliques, and every s-club is also an $(s+1)$-club by definition. Notice the non-hereditary nature of s-clubs, which makes their behavior different from that of cliques for $s \geq 2$: although every subset of a clique is a clique, the same is not true for an s-club. In fact, deciding whether a given s-club is maximal, in the sense that no superset of it is an s-club, is NP-complete for every fixed $s \geq 2$ [28].

Given a graph G and an integer s, the objective of the Max s-Club problem is to compute an s-club of maximum cardinality. We are interested in the exact solution of this problem. Note that the problem becomes trivial if G has diameter at most s.

Max s-Club is NP-hard for every fixed s, even on graphs of diameter $s + 1$ [7]. It remains NP-hard on bipartite graphs for every fixed $s \geq 3$, and on chordal graphs for every even fixed $s \geq 2$ [4]. Max 2-Club is NP-hard on graphs that become

☆ A preliminary version of this paper appeared as an extended abstract in the proceedings of 8th International Conference on Algorithms and Complexity (CIAC 2013).

* Corresponding author. Tel.: +47 93611971.
E-mail addresses: Petr.Golovach@ii.uib.no (P.A. Golovach), pinar.heggernes@ii.uib.no (P. Heggernes), kratsch@univ-metz.fr (D. Kratsch), arashr@sfu.ca (A. Rafiey).

http://dx.doi.org/10.1016/j.dam.2014.04.016
0166-218X/

bipartite by deleting one vertex, on graphs that can be covered by three cliques, and on graphs with domination number two and diameter three [20]. On split graphs Max 2-Club is NP-hard [4], whereas Max s-Club has a trivial solution for all input $s \geq 3$. On general graphs, the problem is fixed-parameter tractable when parameterized by the solution size [13] for every fixed $s \geq 2$, or by the dual of the solution size [31] for every fixed s. Fixed-parameter tractability of Max 2-Club has been studied also with respect to various other parameters [21,20]. Furthermore, Max s-Club can be solved by an $O(1.62^n)$-time algorithm [13]. The problem can be solved in polynomial time on trees and interval graphs for all input values of s, and on graphs of bounded treewidth and graphs of bounded clique-width for every fixed s that is not a part of the input [30].

In this paper we show that Max s-Club can be solved in polynomial time for all odd input values of s on weakly chordal graphs. For subclasses of weakly chordal graphs, we show that the problem can be solved in polynomial time for all input values of s on chordal bipartite graphs, strongly chordal graphs, and distance hereditary graphs. To complement these positive results, we show that on 4-chordal graphs, which form a superclass of weakly chordal graphs, the problem is NP-hard for every fixed s. In addition to these results, we show that the problem is solvable in polynomial time for all input $s \geq 2$ on AT-free graphs. Interestingly, Max Clique is NP-hard on this graph class. The inclusion relationship among the graph classes mentioned above is illustrated in Fig. 1, which also summarizes our results.

2. Definitions and first observations

We refer to the textbook by Diestel [17] for any undefined graph terminology. We consider finite undirected graphs without loops or multiple edges. Such a graph $G = (V, E)$ is identified by its vertex set V and its edge set E. Throughout the paper, we let $n = |V|$ and $m = |E|$. The subgraph of G induced by $U \subseteq V$ is denoted by $G[U]$. For a vertex v, we denote by $N_G(v)$ the set of vertices that are adjacent to v in G. The *distance* $\text{dist}_G(u, v)$ between vertices u and v of G is the number of edges on a shortest path between them. The *diameter* $\text{diam}(G)$ of G is $\max\{\text{dist}_G(u, v) \mid u, v \in V\}$. The *complement* of G is the graph $\overline{G}$ with vertex set V, such that any two distinct vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. For a positive integer k, the *k-th power* G^k of G is the graph with vertex set V, such that any two distinct vertices u, v are adjacent in G^k if and only if $\text{dist}_G(u, v) \leq k$. We say that P is a (u, v)-path if P is a path that joins u and v. The vertices of P different from u and v are the *inner* vertices of P. The *chordality* $ch(G)$ of a graph G is the length of the longest induced cycle in G; if G has no cycles, then $ch(G) = 0$. A set of pairwise adjacent vertices is a *clique*. A clique is *maximal* if no proper superset of it is a clique, and *maximum* if it has maximum size.

For a non-negative integer k, a graph G is *k-chordal* if $ch(G) \leq k$. A graph G is *weakly chordal* if both G and $\overline{G}$ are 4-chordal. A graph is *chordal bipartite* if it is both 4-chordal and bipartite. A graph is *chordal* if it is 3-chordal. A graph is a *split* graph if its vertex set can be partitioned in an independent set and a clique. A chord xy in a cycle C of even length is said to be *odd* if the distance in C between x and y is odd. A graph is *strongly chordal* if it is chordal and every cycle of even length at least 6 has an odd chord. A graph G is a *distance hereditary* if for any connected induced subgraph H of G, if u and v are in H, then $\text{dist}_G(u, v) = \text{dist}_H(u, v)$. An *asteroidal triple (AT)* is a set of three non-adjacent vertices such that between each pair of them there is a path that does not contain a neighbor of the third. A graph is *AT-free* if it contains no AT. Each of these graph classes can be recognized in polynomial (in most cases linear) time and they are closed under taking induced subgraphs [10,18]. See the monographs by Brandstädt et al. [10] and Golumbic [18] for more properties and characterizations of these classes and their inclusion relationships.

Let s be a positive integer. A set of vertices S in G is an *s-club* if $\text{diam}(G[S]) \leq s$. An s-club of maximum size is a *maximum* s-club. Given a graph G and a positive integer s, the Max s-Club problem is to compute a maximum s-club in G. Cliques are exactly 1-clubs, and hence Max 1-Club is equivalent to Max Clique.

Observation 1. *Let G be a graph and let s be a positive integer. If S is an s-club in G then S is a clique in G^s.*

Although Observation 1 is easy to see, it is important to note that the backward implication does not hold in general: a (maximal) clique in G^s is not necessarily an s-club. To see this, let $s = 2$ and consider the graphs shown in Fig. 2(a) and (b). The set of vertices S shown in black in Fig. 2(a) is the unique maximum clique of G_1^2, but clearly S is not a 2-club in G_1, as $G_1[S]$ is not even connected. The example in Fig. 2(b) shows that it does not help to require that $G[S]$ is connected: the set of black vertices S is a maximal clique in G_2^2, $G_2[S]$ is connected, but S is not a 2-club in G_2, because $\text{dist}_{G_2[S]}(u, v) = 3$. Observe also that a maximum s-club in G is not necessarily a maximal clique in G^s. Furthermore, the maximum size of a clique in G_1^2 is strictly greater than the maximum size of a 2-club in G_1. For the set of black vertices S in Fig. 2(b), $S \setminus \{u\}$ and $S \setminus \{v\}$ are maximum 2-clubs, whereas S is a maximal clique in G_2^2.

As we will show in Section 3, for some graph classes, maximal cliques in sth powers are in fact s-clubs. For a positive integer s, we say that a graph class $\mathcal{G}$ has the *s-clique-power* property if for every graph $G \in \mathcal{G}$, every maximal clique in G^s is an s-club in G. Furthermore, we say that $\mathcal{G}$ has the *clique-power* property if every maximal clique in G^s is an s-club in G, for every positive integer s and every graph $G \in \mathcal{G}$. Due to Observation 1, we see that if G belongs to a graph class that has the clique-power property, then a vertex set S in G is a maximal s-club if and only if S is a maximal clique in G^s. As G^s can be computed in time $O(n^3)$ for any positive s, the following is immediate, and it will be the framework in which we obtain our results.

Proposition 1. *Let $\mathcal{G}$ be a graph class that has the clique-power property and let s be a positive integer.*

- *If* Max Clique *can be solved in time $O(f(n))$ on $\{G^s \mid G \in \mathcal{G}\}$, then* Max s-Club *can be solved in time $O(f(n) + n^3)$ on $\mathcal{G}$.*
- *If* Max Clique *is NP-hard on $\{G^s \mid G \in \mathcal{G}\}$, then* Max s-Club *is NP-hard on $\mathcal{G}$.*