Discrete Applied Mathematics 174 (2014) 66–72

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Optimization problems in dotted interval graphs[☆]

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ARTICLE INFO

Article history:
Received 12 December 2012
Received in revised form 16 December 2013
Accepted 17 April 2014
Available online 1 May 2014

Keywords:
Dotted interval graphs
Independent set
Vertex cover
Dominating set
Deletion problems

ABSTRACT

The class of *D-dotted interval* (d-DI) graphs is the class of intersection graphs of arithmetic progressions with jump (common difference) at most d. We consider various classical graph-theoretic optimization problems in d-DI graphs of arbitrarily, but fixed, d.

We show that Maximum Independent Set, Minimum Vertex Cover, and Minimum Dominating Set can be solved in polynomial time in this graph class, answering an open question posed by Jiang (2006). We also show that Minimum Vertex Cover can be approximated within a factor of $(1 + \varepsilon)$, for any $\varepsilon > 0$, in linear time. This algorithm generalizes to a wide class of deletion problems including the classical Minimum Feedback Vertex Set and Minimum Planar Deletion problems.

Our algorithms are based on classical results in algorithmic graph theory and new structural properties of d-DI graphs that may be of independent interest.

1. Introduction

A *dotted interval* $I(s, t, j)$ is an arithmetic progression $\{s, s + j, s + 2j, \dots, t\}$, where s, t and j are positive integers, and the *jump* j divides $t - s$. When $j = 1$, the dotted interval $I(s, t, j)$ is simply the interval $[s, t]$ over the positive integer line. This paper is mainly concerned with dotted interval graphs. A *dotted interval graph* is an intersection graph of dotted intervals. Each vertex v is associated with a dotted interval I_v and we have an edge (u, v) if $I_u \cap I_v \neq \emptyset$. If the jumps of all intervals are at most d, we call the graph *d-dotted-interval* or d-DI for short. See Fig. 1 for an example.

Dotted interval graphs were introduced by Aumann et al. [2] in the context of high throughput genotyping. They used dotted intervals to model microsatellite polymorphisms which are used in a genotyping technique called microsatellite genotyping. The respective genotyping problem translates to Minimum Coloring in d-DI graphs of small d. Aumann et al. [2] showed that Minimum Coloring in d-DI graphs is NP-hard even for $d = 2$. They also provided a $\frac{3}{2}$-approximation algorithm for Minimum Coloring in 2-DI graphs, and a $(\frac{7d}{8} + \frac{3}{8})$-approximation algorithm for general fixed $d \geq 2$. This algorithm was later improved by Jiang [20], and subsequently also by Yanovski [24]. The current best approximation ratio for Minimum Coloring is $\frac{2d+4}{3}$ [24].

Since a dotted interval with jump 1 is a regular interval, dotted interval graphs form natural generalizations of the well-studied class of interval graphs. Interval graphs have been extensively researched in the graph-theoretic community, in particular from the algorithmic viewpoint, because many real-life problems translate to classical graph-theoretic problems

[☆] A preliminary version of this paper appeared in the proceedings of the 38th International Workshop on Graph-Theoretic Concepts in Computer Science (WG), 2012.

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http://dx.doi.org/10.1016/j.dam.2014.04.014

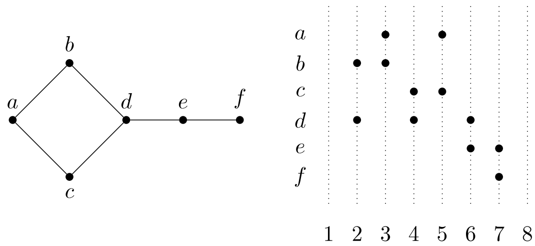

Fig. 1. Example of a 2-DI graph: on the right we have the 2-DI representation of the graph on the left. Notice that the graph is clearly not an interval graph since it contains a hole of length 4.

in interval graphs, and because its rich structure allows in many cases designing efficient algorithms for these problems. Substantial research effort has been devoted to generalizing such algorithms to larger classes of graphs. Examples include algorithms proposed for circular arc graphs [16,18], disc graphs [12,19,22,23], rectangle graphs [1,5,10], k-gap interval graphs [14], multiple-interval graphs [4,9,13], and multiple-subtree graphs [17].

In this paper we study the computational complexity of classical graph-theoretic optimization problems in d-DI graphs. Note that as any graph G is a d-DI graph for large enough d [2], we are interested in studying d-DI graphs for small d; more precisely, we assume $d = O(1)$. Apart from the Minimum Coloring problem, Aumann et al. [2] also considered the Maximum Clique problem in d-DI graphs, and showed that this problem is fixed parameter tractable with respect to d. Jiang [20] studied the problem of Maximum Independent Set in d-DI graphs. He presented a simple $\frac{3}{2}$-approximation algorithm for 2-DI graphs, and a $(\frac{5d}{6} + O(\log d))$-approximation algorithm for d-DI graphs. The question of whether Maximum Independent Set in d-DI graphs, for constant d, is NP-hard was left open by Jiang. He also pointed out that the complexity of other classical graph theoretical problems, such as Minimum Vertex Cover and Minimum Dominating Set, remains open in d-DI graphs.

In this paper we focus mainly on three classical graph-theoretic optimization problems: Maximum Independent Set, Minimum Dominating Set, and Minimum Vertex Cover. We present an $O(dn^d)$-time algorithm for Maximum Independent Set and Minimum Vertex Cover in d-DI graphs with n vertices, and give an $O(d^2 n^{O(d^2)})$-time algorithm for Minimum Dominating Set in d-DI graphs. Thus, we show that these problems are polynomial-time solvable in d-DI graphs for fixed d. It is interesting to note that a similar situation occurs in circular-arc graphs, which also generalize interval graphs, where the unweighted versions of Maximum Independent Set, Minimum Vertex Cover, and Minimum Dominating Set can be solved in linear time [18] and Minimum Coloring is NP-hard [15]. (However, Aumann et al. [2] show that there is a 2-DI graph that is not a circular arc graph, and that for every $d \geq 1$, there is a circular arc graph that is not a d-DI graph.) We also present a linear-time $(1 + \varepsilon)$-approximation algorithm for Minimum Vertex Cover in d-DI graphs. This algorithm can be generalized to a wide range of deletion problems which include among many the classical Minimum Feedback Vertex Set and Minimum Planar Deletion problems. We assume that the d-DI representation of the input graph is given to us.

2. Preliminaries

2.1. Definitions and notation

For $i, k \in \mathbb{Z}$ such that $i < k$, we define $[i, k] := \{i, i+1, \ldots, k-1, k\}$.

Given a dotted interval $I = \{s, s+d, s+2d, \ldots, t\}$, we denote its starting and finishing points by $s(I)$ and $t(I)$, respectively. The jump of I is denoted by $j(I)$, and the *offset* of I is defined as $o(I) := s(I) \bmod d(I)$.

Given a set of dotted intervals $\mathcal{I} = \{I_1, \ldots, I_n\}$, we assume that the intervals are ordered by the starting point, namely that $s(I_i) \leq s(I_{i+1})$, for every i. Dotted intervals with the same starting point are ordered according to their finishing points. Given a dotted interval I_i, we define $\mathcal{I}_{<i} := \{I_j : j < i\}$. Given a point p, and a set of dotted intervals $\mathcal{S} \subseteq \mathcal{I}$, let $\mathcal{S}_p \subseteq \mathcal{S}$ contain the dotted intervals that start at or before p and end at or after p, namely $\mathcal{S}_p := \{I \in \mathcal{S} : p \in [s(I), t(I)]\}$. (Note that it is possible $I \in \mathcal{S}_p$ and $p \notin I$.)

Given an undirected graph $G = (V, E)$ we define $n := |V|$. Also, for any subset $A \subseteq V$, we use

$$G[A] = (A, \{(u, v) \in E : u, v \in A\})$$

to denote the graph induced by A. Let $w : V \to \mathbb{R}^+$ be a vertex weight function; for any $A \subseteq V$, we use the shorthand notation $w(A) = \sum_{u \in A} w(u)$. A subset $A \subseteq V$ is said to be *independent* if no two vertices in A are connected by an edge in E, i.e., if $E(G[A]) = \emptyset$; the Maximum Independent Set problem is to find an independent set of maximum weight. A subset $A \subseteq V$ is said to be a *vertex cover* if every edge in E has at least one endpoint in A, namely if $E(G[V \setminus A]) = \emptyset$; the Minimum Vertex Cover is to find a vertex cover of minimum weight. A subset $A \subseteq V$ is said to be *dominating* if every vertex $v \subseteq V \setminus A$ has at least one neighbor in A; the Minimum Dominating Set problem is to find a dominating set of minimum weight.

2.2. Simple observations

Let $\mathcal{I}$ be a representation of a d-DI graph G, and denote $\ell(d) = \operatorname{lcm}\{2, \ldots, d\}$, the least common multiple of the numbers $2, \ldots, d$.