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Pivotal decompositions of functions

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ABSTRACT

We extend the well-known Shannon decomposition of Boolean functions to more general classes of functions. Such decompositions, which we call pivotal decompositions, express the fact that every unary section of a function only depends upon its values at two given elements. Pivotal decompositions appear to hold for various function classes, such as the class of lattice polynomial functions or the class of multilinear polynomial functions. We also define function classes characterized by pivotal decompositions and function classes characterized by their unary members and investigate links between these two concepts.

1. Introduction

A remarkable (though immediate) property of Boolean functions is the so-called *Shannon decomposition*, or *Shannon expansion* (see [20]), also called *pivotal decomposition* [2]. This property states that, for every Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ and every $k \in [n] = \{1,\ldots,n\}$, the following decomposition formula holds:

$$f(\mathbf{x}) = x_k\, f(\mathbf{x}_k^1) + \overline{x}_k\, f(\mathbf{x}_k^0)\,, \quad \mathbf{x} = (x_1,\ldots,x_n) \in \{0,1\}^n, \tag{1}$$

where $\overline{x}_k = 1 - x_k$ and $\mathbf{x}_k^a$ is the n-tuple whose ith coordinate is a, if $i = k$, and x_i, otherwise. Here the '+' sign represents the classical addition for real numbers.

Decomposition formula (1) means that we can precompute the function values for $x_k = 0$ and $x_k = 1$ and then select the appropriate value depending on the value of x_k. By analogy with the cofactor expansion formula for determinants, here $f(\mathbf{x}_k^1)$ (resp. $f(\mathbf{x}_k^0)$) is called the cofactor of x_k (resp. $\overline{x}_k$) for f and is derived by setting $x_k = 1$ (resp. $x_k = 0$) in f.

Clearly, the addition operation in (1) can be replaced with the maximum operation $\vee$, thus yielding the following alternative formulation of (1):

$$f(\mathbf{x}) = (x_k\, f(\mathbf{x}_k^1)) \vee (\overline{x}_k\, f(\mathbf{x}_k^0))\,, \quad \mathbf{x} \in \{0,1\}^n,\ k \in [n].$$

Equivalently, (1) can also be put in the form

$$f(\mathbf{x}) = (x_k \vee f(\mathbf{x}_k^0))\,(\overline{x}_k \vee f(\mathbf{x}_k^1))\,, \quad \mathbf{x} \in \{0,1\}^n,\ k \in [n]. \tag{2}$$

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As it is well known, repeated applications of (1) show that any n-ary Boolean function can always be expressed as the multilinear polynomial function

$$f(\mathbf{x}) = \sum_{S\subseteq[n]} f(\mathbf{1}_S) \prod_{i\in S} x_i \prod_{i\in[n]\setminus S} \bar{x}_i\,, \quad \mathbf{x}\in\{0,1\}^n, \tag{3}$$

where $\mathbf{1}_S$ is the characteristic vector of S in $\{0,1\}^n$, that is, the n-tuple whose ith coordinate is 1, if $i \in S$, and 0, otherwise.

If f is nondecreasing (i.e., the map $z \mapsto f(\mathbf{x}_k^z)$ is isotone for every $\mathbf{x} \in \{0,1\}^n$ and every $k \in [n]$), then by expanding (2) we see that the decomposition formula reduces to

$$f(\mathbf{x}) = (x_k f(\mathbf{x}_k^1)) \vee f(\mathbf{x}_k^0)\,, \quad \mathbf{x}\in\{0,1\}^n,\ k\in[n], \tag{4}$$

or, equivalently,

$$f(\mathbf{x}) = \mathrm{med}(x_k, f(\mathbf{x}_k^1), f(\mathbf{x}_k^0))\,, \quad \mathbf{x}\in\{0,1\}^n,\ k\in[n], \tag{5}$$

where med is the ternary median operation defined by

$$\mathrm{med}(x,y,z) = (x\wedge y)\vee(y\wedge z)\vee(z\wedge x)$$

and $\wedge$ is the minimum operation.

Interestingly, the following decomposition formula also holds for nondecreasing n-ary Boolean functions:

$$f(\mathbf{x}) = x_k\,(f(\mathbf{x}_k^1)\vee f(\mathbf{x}_k^0)) + \bar{x}_k\,(f(\mathbf{x}_k^1)\wedge f(\mathbf{x}_k^0))\,, \quad \mathbf{x}\in\{0,1\}^n,\ k\in[n]. \tag{6}$$

Actually, any of the decomposition formulas (4)–(6) exactly expresses the fact that f should be nondecreasing and hence characterizes the subclass of nondecreasing n-ary Boolean functions. We state this result as follows.

Proposition 1.1. *A Boolean function $f\colon\{0,1\}^n \to \{0,1\}$ is nondecreasing if and only if it satisfies any of the decomposition formulas (4)–(6).*

Decomposition property (1) also holds for functions $f\colon\{0,1\}^n \to \mathbb{R}$, called *$n$-ary pseudo-Boolean functions*. As a consequence, these functions also have the representation given in (3). Moreover, formula (6) clearly characterizes the subclass of nondecreasing n-ary pseudo-Boolean functions.

The *multilinear extension* of a pseudo-Boolean function $f\colon\{0,1\}^n \to \mathbb{R}$ is the function $\hat{f}\colon[0,1]^n \to \mathbb{R}$ defined by (see Owen [16,17])

$$\hat{f}(\mathbf{x}) = \sum_{S\subseteq[n]} f(\mathbf{1}_S) \prod_{i\in S} x_i \prod_{i\in[n]\setminus S} (1-x_i)\,, \quad \mathbf{x}\in[0,1]^n. \tag{7}$$

Actually, a function is the multilinear extension of a pseudo-Boolean function if and only if it is a multilinear polynomial function, i.e., a polynomial function of degree $\leqslant 1$ in each variable. Thus defined, one can easily see that the class of multilinear polynomial functions can be characterized as follows.

Proposition 1.2. *A function $f\colon[0,1]^n \to \mathbb{R}$ is a multilinear polynomial function if and only if it satisfies*

$$f(\mathbf{x}) = x_k f(\mathbf{x}_k^1) + (1-x_k) f(\mathbf{x}_k^0)\,, \quad \mathbf{x}\in[0,1]^n,\ k\in[n]. \tag{8}$$

Interestingly, Eq. (8) provides an immediate proof of the property

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = f(\mathbf{x}_k^1) - f(\mathbf{x}_k^0),$$

which holds for every multilinear polynomial function $f\colon[0,1]^n \to \mathbb{R}$.

As far as nondecreasing multilinear polynomial functions are concerned, we have the following characterization, which is a special case of Corollary 4.8. Recall first that a multilinear polynomial function is nondecreasing if and only if so is its restriction to $\{0,1\}^n$ (i.e., its defining pseudo-Boolean function).

Proposition 1.3. *A function $f\colon[0,1]^n \to \mathbb{R}$ is a nondecreasing multilinear polynomial function if and only if it satisfies*

$$f(\mathbf{x}) = x_k\,(f(\mathbf{x}_k^1)\vee f(\mathbf{x}_k^0)) + \bar{x}_k\,(f(\mathbf{x}_k^1)\wedge f(\mathbf{x}_k^0))\,, \quad \mathbf{x}\in[0,1]^n,\ k\in[n]. \tag{9}$$

The decomposition formulas considered in this introduction share an interesting common feature, namely the fact that any variable, here denoted x_k and called *pivot*, can be pulled out of the function, reducing the evaluation of $f(\mathbf{x})$ to the