Discrete Applied Mathematics 174 (2014) 113–121

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Set graphs. IV. Further connections with claw-freeness

Martin Milanič [a,b], Alexandru I. Tomescu [c,*]

[a] University of Primorska, UP IAM, Muzejski trg 2, SI6000 Koper, Slovenia
[b] University of Primorska, UP FAMNIT, Glagoljaška 8, SI6000 Koper, Slovenia
[c] Helsinki Institute for Information Technology HIIT, Department of Computer Science, University of Helsinki, Finland

ARTICLE INFO

Article history:
Received 26 August 2013
Received in revised form 23 April 2014
Accepted 1 May 2014
Available online 16 May 2014

Keywords:
Acyclic orientation
Extensionality
Claw-free graph
Hereditary graph class
Set graph

ABSTRACT

A graph G is said to be a *set graph* if it admits an acyclic orientation that is also *extensional*, in the sense that the out-neighborhoods of its vertices are pairwise distinct. Equivalently, set graphs are the underlying graphs of digraph representations of hereditarily finite sets, where a set is *hereditarily finite* if it is finite and has only hereditarily finite sets as members.

It is known that every connected $K_{1,3}$-free (or *claw-free*) graph is a set graph, and that an extensional acyclic orientation of such a graph can be found in polynomial time. In this paper, we generalize this result in three different directions. First, we identify the largest hereditary class of graphs G such that for every connected induced subgraph H of G, it holds that H is a set graph if and only if it is claw-free. Second, we consider graphs in which no two distinct induced claws have equal or adjacent centers, and prove that in this class of graphs set graphs can be equivalently characterized in terms of a property related to successive vertex removal. Finally, we show that for every $r \geqslant 1$, connected $K_{1,r+2}$-free graphs admit an acyclic orientation that is *r-extensional*, in the sense that at most r distinct vertices can have the same out-neighborhood. This also leads to a simple linear time algorithm for finding an extensional acyclic orientation of a given connected claw-free graph, thus improving over the previous algorithm.

1. Introduction

An *extensional* acyclic digraph is an acyclic digraph with the property that its vertices have pairwise distinct sets of out-neighbors. A *hereditarily finite set* is a finite set such that all its members are hereditarily finite sets. Extensional acyclic digraphs model hereditarily finite sets; see Fig. 1 for an example.

Extensional acyclic digraphs came into the spotlight with the rise of the field of computable set theory [5,19], which aims to discover decidable fragments of set theory, and efficient decidability algorithms. This has led to programming languages such as SETL [25], {log} [7], CLP($\mathcal{SET}$) [8], and to an automatic proof verifier Referee [16,24], all having hereditarily finite sets as built-in concepts. On the more practical side, extensional acyclic digraphs can be characterized in terms of separating codes in digraphs [9,6,13], with applications in the design of emergency sensor networks in facilities, or in fault detection in multiprocessor systems. Some enumerative results related to extensional acyclic digraphs are [20,21,29,23,22]. See [14,13,21] for a more detailed discussion and further references.

In [14] we started a study of extensional acyclic digraphs and their underlying graphs – called *set graphs* – which we continue in this paper. A sufficient (though not necessary) condition for a graph to be a set graph is to be connected and claw-free.

* Corresponding author.
E-mail addresses: martin.milanic@upr.si (M. Milanič), alexandru.tomescu@gmail.com, tomescu@cs.helsinki.fi (A.I. Tomescu).

http://dx.doi.org/10.1016/j.dam.2014.05.011
0166-218X/

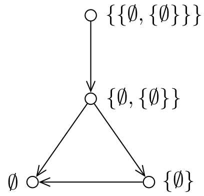

Fig. 1. The extensional acyclic digraph D_x modeling the hereditarily finite set $x = \{\{\{\emptyset,\{\emptyset\}\}\}\}$; D_x is the digraph $(\mathsf{TrCl}(x), \{(y,z) \mid z \in y \in \mathsf{TrCl}(x)\})$, where $\mathsf{TrCl}(x)$, called the *transitive closure* of x, is the set recursively defined as $x \cup \bigcup_{y \in x} \mathsf{TrCl}(y)$. The underlying graph of D_x is the *paw*.

Theorem 1 (*[14]*)**.** *Every connected claw-free graph is a set graph. An extensional acyclic orientation of a connected claw-free graph can be found in polynomial time.*

This paper is dedicated to generalizing Theorem 1 in three different directions. Before describing these generalizations, let us first note that Theorem 1 was a fruitful result, since, on one hand, it gave a hint for simpler proofs [27,14] of the well-known properties that squares of connected claw-free graphs are vertex pancyclic [12], and that connected claw-free graphs of even order have a perfect matching [26,28]. On the other hand, since one of our motivations for the study of set graphs is to identify which graphs can be compactly represented by a hereditarily finite set,[1] we took advantage of the representability of connected claw-free graphs by hereditarily finite sets implied by Theorem 1 to formalize these two simpler proofs in the automatic proof verifier Referee [18]; however, the proof in [14] of Theorem 1 itself is not simple enough to be easily formalized in Referee.

The results in this paper are motivated also by the fact that one cannot hope for a 'good' characterization of set graphs, since the recognition problem for set graphs is in general NP-complete [13]. The proof of this fact uses the following result, where $S(G)$ denotes the total subdivision graph of G, that is, the graph obtained from G by subdividing each edge once.

Theorem 2 (*[13]*)**.** *A graph G has a Hamiltonian path if and only if $S(G)$ is a set graph.*

The following two conditions for set graphs, one sufficient and one necessary, were also established in [14].

Theorem 3 (*[14]*)**.** *If G has a Hamiltonian path, then G is a set graph.*

Lemma 1 (*[14]*)**.** *If G is a set graph, then for every $X \subseteq V(G)$, $G - X$ has at most $2^{|X|}$ connected components.*

In particular, the condition stated in Lemma 1 implies that every set graph G satisfies the following conditions:

(1) G is connected, and
(2) for every vertex v of G, the graph $G - v$ has at most two connected components.

In what follows, we will refer to condition (2) above as the *cut vertex condition*. The claw $K_{1,3}$ violates the cut vertex condition, hence is not a set graph. However, it is the unique minimal connected non-set graph, as shown by Theorem 1.

The three generalizations of Theorem 1 are as follows.

In Section 2 we characterize, in terms of forbidden induced subgraphs, the class of graphs G such that for every connected induced subgraph H of G, it holds that H is a set graph if and only if it is claw-free. This class of graphs is a common generalization of block graphs and claw-free graphs. Moreover, it turns out that this class of graphs coincides with the class of graphs G such that for every induced subgraph H of G, it holds that H is a set graph if and only if it satisfies the cut vertex condition. This implies that the recognition of set graphs in this class can be carried out in linear time. We also give a structural result describing a recursive way of building every graph in this class from claw-free graphs.

In Section 3 we investigate another generalization of claw-free graphs: the so-called *claw restricted graphs*, which we define as graphs in which no two distinct induced claws have equal or adjacent centers. We prove that every set graph is reducible to a complete graph on at most two vertices by successively removing either one vertex, or a set of two vertices consisting of a leaf and its neighbor, in such a way that all the intermediate graphs are connected and satisfy the cut vertex condition. It turns out that for claw restricted graphs, this necessary condition is also sufficient for being a set graph. On the negative side, we show that the set graph recognition remains NP-complete in the class of claw restricted graphs.

In Section 4 we introduce a relaxation of the notion of extensional orientation, the so-called r-extensional orientation. For $r = 1$, this notion coincides with extensional orientation. We generalize Theorem 1 by showing that for every $r \geqslant 1$, every connected $K_{1,r+2}$-free graph admits an r-extensional acyclic orientation with a unique sink. Our proof is constructive and leads to a simple linear time algorithm for finding such an orientation. In particular, this improves on the polynomial time algorithm for finding an extensional acyclic orientation of a connected claw-free graph given by (the proof of) Theorem 1 in [14]. (A straightforward implementation of that algorithm has complexity $O(n^3\Delta^2)$ if the input graph has n vertices and

[1] For example, the paw can be compactly represented by the set $\{\{\{\emptyset,\{\emptyset\}\}\}\}$, cf. Fig. 1; see [14,18] for further details.