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Nearly equal distances in metric spaces☆

Amit Ophir, Rom Pinchasi *

Mathematics Department, Technion—Israel Institute of Technology, Haifa 32000, Israel

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ABSTRACT

Let (X, d) be any finite metric space with n elements. We show that there are two pairs of distinct elements in X that determine two nearly equal distances in the sense that their ratio differs from 1 by at most $\frac{9 \log n}{n^2}$. This bound (apart for the multiplicative constant) is best possible and we construct a metric space that attains this bound.

We discuss related questions and consider in particular the Euclidean metric space.

1. Introduction

Let (X, d) be a finite metric space with $n = |X|$ elements. Consider all the $\binom{n}{2}$ distances between pairs of distinct elements in X. In principle all these distances can be equal as is the case if the metric d is the discrete metric. However, can all those distances be very much different from each other? In Section 2 we prove the following theorem which asserts that there always must be two distances that are nearly equal.

Theorem 1. *Any finite metric space (X, d) of $n \geq 3$ elements contains two pairs (not necessarily disjoint) of distinct elements $\{x, y\}$ and $\{a, b\}$ such that $|\frac{d(x,y)}{d(a,b)} - 1| \leq \frac{9 \log n}{n^2}$.*

The bound in Theorem 1 is best possible for general metric spaces:

Theorem 2. *For every n there exists a metric space (X, d) on n elements such that for every two pairs (not necessarily disjoint) of distinct elements $\{x, y\}$ and $\{a, b\}$ we have $|\frac{d(x,y)}{d(a,b)} - 1| \geq \frac{\log n}{20n^2}$.*

The proof of Theorems 1 and 2 is given in Section 2, which contains also a generalization of Theorem 1 to the case of k nearly equal distances. We note that a different (much weaker) notion of *nearly equal distances* was studied, for example, in [3] (see also [6,7]), where two distances are said to be nearly equal if their difference is at most 1 while every two points in the set are at distance at least 1 from each other. Our notion of being 'nearly equal' is considerably stronger. In [6,7] the relation between the diameter of a set of points and the number of pairs of 'nearly equal' distances (in the weaker sense) is studied and at least philosophically it is somewhat related to the proof of Theorem 1.

Notice that, there are two different kinds of pairs of distances in a metric space. That is, given two distinct pairs of elements $\{x, y\}$ and $\{a, b\}$, either they are disjoint, or they have one element in common. Theorem 1 guarantees two pairs of elements with nearly equal distances. However, it does not tell us anything on whether these two pairs are disjoint or not.

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* Corresponding author.
E-mail addresses: samitoph@t2.technion.ac.il (A. Ophir), room@math.technion.ac.il (R. Pinchasi).

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If we insist on finding two disjoint pairs of elements $\{x, y\}$ and $\{a, b\}$ such that $\frac{d(x,y)}{d(a,b)}$ is close to 1, then in very simple metric spaces we are doomed to fail. Consider for example the metric space $X = \{1, 3, 3^2, 3^3, \dots, 3^{n-1}\}$, as a subset of $\mathbb{R}$ with the ordinary Euclidean metric. Given two disjoint pairs of elements of X, $\{3^x, 3^y\}$ and $\{3^a, 3^b\}$, assume without loss of generality that b is the maximum number among $\{x, y, a, b\}$. Then $d(3^a, 3^b) = 3^b - 3^a \geq 3^b - 3^{b-1} = 2 \cdot 3^{b-1}$ while (assuming without loss of generality that $y > x$) $d(3^y, 3^x) = 3^y - 3^x \leq 3^y \leq 3^{b-1}$. Hence the ratio between $d(3^a, 3^b)$ and $d(3^y, 3^x)$ is at least 2, quite far from 1. It turns out however that one can always find two distinct pairs of elements in X that are not disjoint with nearly equal distances:

Theorem 3. *Let (X, d) be a metric space with n elements. There exist three distinct elements x, a, b in X such that* $|\frac{d(x,a)}{d(x,b)} - 1| \leq \frac{3}{n}$.

Here too we can show by construction that the bound in Theorem 3 is best possible:

Theorem 4. *For every n there exists a metric space (X, d) such that for every three distinct elements x, a, b in X we have* $|\frac{d(x,a)}{d(x,b)} - 1| \geq \frac{1}{2n}$.

The proofs of Theorems 3 and 4 are presented in Section 3 together with a generalization of Theorem 3 to the case of k nearly equal distances from the same point. Section 4 contains some discussion in the specific case of subsets of the Euclidean metric space $\mathbb{R}$, where there are some very interesting open problems. In Section 5 we bring some consequences of the above theorems to the Euclidean space as well as more open problems.

2. The existence of nearly equal distances in a finite metric space

We start with the proof of Theorem 1:

Let $x, y \in X$ be a pair of distinct elements in X such that $d(x, y)$ is minimum. Without loss of generality we may assume that $d(x, y) = 1$, for otherwise we replace the metric $d(\cdot, \cdot)$ by the metric $\frac{1}{d(x,y)} d(\cdot, \cdot)$.

Let $1 = d_1 \leq d_2 \leq \cdots \leq d_{\binom{n}{2}}$ denote all the distances between pairs of distinct elements in X listed in an increasing order. In fact we may assume strict inequalities between these distances for otherwise we are done.

If we assume by contradiction that for every $1 \leq i < \binom{n}{2}$ we have $\frac{d_{i+1}}{d_i} \geq 1 + \frac{9 \log n}{n^2}$, then $d_{\binom{n}{2}} \geq (1 + 9\frac{\log n}{n^2})^{\binom{n}{2}-1} \geq e^{\frac{9}{3}\log n} = n^3$ (here we assume $n \geq 7$). It follows that there is at least one element $z \in X$ such that $d(x, z) \geq \frac{1}{2} n^3$. Therefore, we have

$$\left|\frac{d(y,z)}{d(x,z)} - 1\right| = \left|\frac{d(y,z) - d(x,z)}{d(x,z)}\right| \leq \frac{d(x,y)}{d(x,z)} \leq \frac{2}{n^3}. \quad \blacksquare$$

In [5] Erdős and Turán show that there are sets of cn integers between 1 and n^2 with pairwise distinct differences, where $c > 0$ is an absolute constant that can be taken to be arbitrary close to $\frac{1}{\sqrt{2}}$ for large enough n. These sets are sometimes called *Sidon sets* after Simon Sidon who initiated their study.

Consider such a set X of integers as a finite metric space with the ordinary Euclidean metric induced from $\mathbb{R}$. Since any two distances among pairs of numbers from the set X differ by at least 1 (because they are distinct integers), we have for every two distinct pairs $\{x, y\}$ and $\{a, b\}$ from X:

$$\left|\frac{d(a,b)}{d(x,y)} - 1\right| = \left|\frac{d(a,b) - d(x,y)}{d(x,y)}\right| \geq \frac{1}{d(x,y)} \geq \frac{1}{n^2}.$$

This shows that apart from the $\log n$ term the bound in Theorem 1 is best possible already in subsets of $\mathbb{R}$ with the ordinary Euclidean metric.

Can we do better than Sidon sets in $\mathbb{R}$? This question is open and we state it as a problem:

Problem A. Do there exist sets X of n real numbers such that for every two distinct pairs $\{x, y\}$ and $\{a, b\}$ from X we have $|\frac{d(a,b)}{d(x,y)} - 1| = \frac{\omega(n)}{n^2}$, where $\omega(n)$ is a function that goes to infinity with n.

We shall now prove Theorem 2 and show by construction that there are metric spaces in which the answer to the analogous problem to Problem A is positive.

Proof of Theorem 2. We will assume that n is large enough ($n > 100$ should be enough). We define the following metric $d(\cdot, \cdot)$ on the elements $x_1, \dots, x_n$: For $i = j$ we define $d(x_i, x_i) = 0$ and for $i < j$ we define $d(x_i, x_j) = d(x_j, x_i) = (1 + \frac{\log n}{10n^2})^{\binom{j+1}{2} - i}$. We first show that this is indeed a metric. We observe that for $1 \leq i < j$ we have $\binom{j}{2} < \binom{j+1}{2} - i < \binom{j+1}{2}$. Therefore, if $i < j$ and $a < b$ are all integers and $\binom{j+1}{2} - i = \binom{b+1}{2} - a$, then $j = b$ and $i = a$. Notice also that if $i < j$ and $a < b$ and $j > b$, then $d(x_i, x_j) \geq \binom{j}{2} \geq \binom{b+1}{2} \geq d(x_a, x_b)$.

Hence, in order to show that d satisfies the triangle inequality, then given $i < j < k$ it is enough to show that $d(x_i, x_k) \leq d(x_i, x_j) + d(x_j, x_k)$, as $d(x_i, x_k)$ is the largest among the three mutual distances between x_i, x_j, and x_k.