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Note

Domination and total domination in cubic graphs of large girth

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ABSTRACT

The domination number $\gamma(G)$ and the total domination number $\gamma_t(G)$ of a graph G without an isolated vertex are among the most well-studied parameters in graph theory. While the inequality $\gamma_t(G) \leq 2\gamma(G)$ is an almost immediate consequence of the definition, the extremal graphs for this inequality are not well understood. Furthermore, even very strong additional assumptions do not allow us to improve the inequality by much.

In the present paper we consider the relation of $\gamma(G)$ and $\gamma_t(G)$ for cubic graphs G of large girth. Clearly, in this case $\gamma(G)$ is at least $n(G)/4$ where $n(G)$ is the order of G. If $\gamma(G)$ is close to $n(G)/4$, then this forces a certain structure within G. We exploit this structure and prove an upper bound on $\gamma_t(G)$, which depends on the value of $\gamma(G)$. As a consequence, we can considerably improve the inequality $\gamma_t(G) \leq 2\gamma(G)$ for cubic graphs of large girth.

1. Introduction

For a finite, simple, and undirected graph G, a set D of vertices of G is a *dominating set of* G if every vertex in $V(G) \setminus D$ has a neighbor in D. Similarly, a set T of vertices of G is a *total dominating set of* G if every vertex in $V(G)$ has a neighbor in T. Note that a graph has a total dominating set exactly if it has no isolated vertex. The minimum cardinalities of a dominating and a total dominating set of G are known as the *domination number* $\gamma(G)$ *of* G and the *total domination number* $\gamma_t(G)$ *of* G, respectively. These two parameters are among the most fundamental and well-studied parameters in graph theory [5,6,8]. In view of their computational hardness especially upper bounds were investigated in great detail.

The two parameters are related by some very simple inequalities. Let G be a graph without isolated vertices. Since every total dominating set of G is also a dominating set of G, we immediately obtain

$$\gamma_t(G) \geq \gamma(G). \tag{1}$$

Similarly, if D is a dominating set of G, then adding, for every isolated vertex u of the subgraph $G[D]$ of G induced by D, a neighbor of u in G to the set D results in a total dominating set of G, which implies

$$\gamma_t(G) \leq 2\gamma(G). \tag{2}$$

The complete bipartite graph $K_{n/2,n/2}$ and the complete graph K_n show that (1) and (2) are sharp, respectively. In [4,7] the trees that satisfy (1) or (2) with equality are characterized constructively.

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While numerous very deep results concerning bounds on the domination number and the total domination number under various conditions have been obtained, the relation of these two parameters is not really well understood. The characterization of the extremal graphs for (1) and (2) and/or improvements of (1) and (2) even under strong additional assumptions appear to be very difficult. If the graph G arises, for instance, by subdividing every edge of the complete graph K_n with $n \geq 3$ twice, then $\gamma(G) = n$ and $\gamma_t(G) = 2n-1$, that is, forbidding cycles of length up to 8 does not allow us to improve (2) by much. For a positive integer k, let $[k]$ denote the set $\{1, 2, \ldots, k\}$. If the graph G has vertex set $\bigcup_{i\in[k]}(A_i \cup B_i \cup C_i)$, where

- the sets A_i, B_i, and C_i for all $i \in [k]$ are disjoint,
- $|A_i| = a$, $|B_i| = a + 1$, and $|C_i| = ka$ for every $i \in [k]$ and some $a \in \mathbb{N}$,
- the closed neighborhood $N_G[u]$ of a vertex u in A_j for $j \in [k]$ is $B_j \cup \bigcup_{i\in[k]} A_i$,
- the closed neighborhood $N_G[v]$ of a vertex v in B_j for $j \in [k]$ is $A_j \cup \{v\} \cup C_j$, and
- the closed neighborhood $N_G[w]$ of a vertex w in C_j for $j \in [k]$ is $B_j \cup C_j$,

then G is regular of degree $(k+1)a$, has connectivity a, diameter 5, $\gamma(G) = k + 1$, and $\gamma_t(G) = 2k$, that is, a large minimum degree, a large degree of regularity, a large connectivity, a small diameter, and a large value of the domination number do not force any serious improvement of (2).

In the present paper we consider the relation between the domination number and the total domination number for cubic graphs of large girth.

Let G be a cubic graph of order n and girth at least g, that is, G has no cycles of length less than g. Clearly, $\gamma(G) \geq \frac{1}{4}n$ and $\gamma_t(G) \geq \frac{1}{3}n$. The best published upper bound on the domination number of G, improving earlier results from [13,14], is due to Král' et al. [12], who show

$$\gamma(G) \leq 0.299871n + O\left(\frac{n}{g}\right). \tag{3}$$

Combining this with $\gamma_t(G) \geq \frac{1}{3}n$, we obtain the following improvement of (1).

Corollary 1. *If G is a cubic graph of order n and girth at least g, then*

$$\frac{\gamma_t(G)}{\gamma(G)} \geq 1.111589 - O\left(\frac{1}{g}\right).$$

In a recent preprint [11] Hoppen and Wormald improve (3) further to $\gamma(G) \leq 0.27942n + O\left(\frac{n}{g}\right)$, which improves the bound in Corollary 1 to $\frac{\gamma_t(G)}{\gamma(G)} \geq 1.192947 - O\left(\frac{1}{g}\right)$.

For a graph G of order n, minimum degree at least 2, and girth at least g, Henning and Yeo [9,10] show $\gamma_t(G) \leq \frac{1}{2}n + O\left(\frac{n}{g}\right)$. Applying a trick from [13], this result leads to the following corollary. Recall that the line graph of a graph G has vertex set $E(G)$ and edge set $\{ef : e, f \in E(G) \text{ and } |e \cap f| = 1\}$. Furthermore, the kth power of a graph G has vertex set $V(G)$ and edge set $\{uv : u, v \in V(G) \text{ and } 0 < \text{dist}_G(u, v) \leq k\}$.

Corollary 2. *If G is a cubic graph of order n and girth at least g, then*

$$\gamma_t(G) \leq \frac{121}{248}n + O\left(\frac{n}{g}\right) \leq 0.488n + O\left(\frac{n}{g}\right). \tag{4}$$

Proof. Let G be as in the statement. In view of the desired bound, we may assume that g is sufficiently large. Since the 5th power of the line graph of G is neither an odd cycle nor complete, has order $\frac{3}{2}n$, and maximum degree 124, the theorem of Brooks [3] implies that there is a set M of at least $\frac{3}{248}n$ edges of G such that for every two vertices u and v that are incident with distinct edges in M, we have $\text{dist}_G(u, v) \geq 5$. Let T_0 denote the set of $2|M|$ vertices incident with the edges in M and let $G_1 = G \setminus N_G[T_0]$. By construction, the graph G_1 has order $n - 6|M|$, minimum degree at least 2, and girth at least g. By the above result of Henning and Yeo, the graph G_1 has a total dominating set T_1 of order at most $\frac{1}{2}(n - 6|M|) + O\left(\frac{n}{g}\right)$. Since $T_0 \cup T_1$ is a total dominating set of G, we obtain

$$\begin{aligned}
\gamma_t(G) &\leq 2|M| + \frac{1}{2}(n - 6|M|) + O\left(\frac{n}{g}\right) \\
&= \frac{1}{2}n - |M| + O\left(\frac{n}{g}\right) \\
&\leq \frac{1}{2}n - \frac{3}{248}n + O\left(\frac{n}{g}\right) \\
&= \frac{121}{248}n + O\left(\frac{n}{g}\right),
\end{aligned}$$

which completes the proof. □