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Note

# Interval edge-colorings of $K_{1,m,n}$

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ABSTRACT

An edge-coloring of a graph $G$ with colors $1, \ldots, t$ is an interval $t$-coloring if all colors are used, and the colors of edges incident to each vertex of $G$ are distinct and form an interval of integers. A graph $G$ is interval colorable if it has an interval $t$-coloring for some positive integer $t$. In this note we prove that $K_{1,m,n}$ is interval colorable if and only if $\gcd(m+1, n+1) = 1$, where $\gcd(m+1, n+1)$ is the greatest common divisor of $m+1$ and $n+1$. It settles in the affirmative, a conjecture of Petrosyan.



## 1. Introduction

All graphs in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph $G$, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d(v)$, the maximum degree of $G$ by $\Delta(G)$ and the edge-chromatic number of $G$ by $\chi'(G)$. We denote by $[a, b]$ the interval of integers $\{a, \ldots, b\}$. The terms, notations and concepts that we do not define can be found in [18].

A *proper edge-coloring* of a graph $G$ is a coloring of the edges of $G$ such that no two adjacent edges receive the same color. If $\alpha$ is a proper coloring of $G$ and $v \in V(G)$, then $S(v, \alpha)$ (*spectrum* of a vertex $v$) denotes the set of colors of edges incident to $v$. A proper edge-coloring of a graph $G$ with colors $1, \ldots, t$ is an *interval $t$-coloring* if all colors are used, and for any vertex $v$ of $G$, the set $S(v, \alpha)$ is an interval of integers. A graph $G$ is *interval colorable* if it has an interval $t$-coloring for some positive integer $t$. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the least and the greatest values of $t$ for which $G$ has an interval $t$-coloring are denoted by $w(G)$ and $W(G)$, respectively.

The concept of interval edge-coloring was introduced by Asratian and Kamalian [1]. In [1,2], they proved that if $G$ is interval colorable, then $\chi'(G) = \Delta(G)$. General bounds on $w(G)$ and $W(G)$ were obtained in [1,2,8,6,10]. Interval edge-colorings of several classes of graphs were investigated, including complete bipartite graphs, trees [9], grids [5,16], planar graphs [3], and $n$-dimensional cubes [12,16]. It is shown that deciding whether a graph has an interval coloring is *NP*-complete even for regular graphs [1,2] and bipartite graphs [17].

Not all the graphs satisfying the condition $\chi'(G) = \Delta(G)$ are interval colorable. Several classes of interval non-colorable bipartite graphs were investigated in [17,7,15]. Recently, Petrosyan [13] has described another class of interval non-colorable graphs:

**Theorem 1.** *If $G$ is an Eulerian graph and $|E(G)|$ is odd, then $G \notin \mathfrak{N}$.*

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Several papers are devoted to the interval edge-colorings of various cases of complete multipartite graphs. In [9] Kamalian obtained the following result [1,2]:

**Theorem 2.** *For any* $m, n \in \mathbb{N}$, $K_{m,n} \in \mathfrak{N}$ *and*
(i) $w(K_{m,n}) = m + n - \gcd(m, n)$
(ii) $W(K_{m,n}) = m + n - 1$
(iii) *if* $w(K_{m,n}) \le t \le W(K_{m,n})$, *then* $K_{m,n}$ *has an interval* $t$*-coloring.*

Also, he showed that complete graphs are interval colorable if and only if the number of vertices is even. Moreover, for any $n \in \mathbb{N}$, $w(K_{2n}) = 2n - 1$. Kamalian could also obtain a lower bound on $W(K_{2n})$, which was later improved by Petrosyan in [12]:

**Theorem 3.** *If* $n = p2^q$, *where* $p$ *is odd and* $q$ *is nonnegative, then*

$$W(K_{2n}) \ge 4n - 2 - p - q.$$

The best known upper bound for $W(K_{2n})$ is $4n - 5$ $(n \ge 3)$ [11] and its exact value is known only for $n \le 4$.

Recently, Petrosyan [14] obtained the following result on interval edge-colorings of complete balanced multipartite graphs:

**Theorem 4.** *If* $K_{n,\dots,n}$ *is a complete balanced* $k$*-partite graph, then* $K_{n,\dots,n} \in \mathfrak{N}$ *if and only if* $nk$ *is even. Moreover, if* $nk$ *is even, then* $w(K_{n,\dots,n}) = n(k-1)$ *and* $W(K_{n,\dots,n}) \ge \left(\frac{3}{2}k - 1\right)n - 1$.

Feng and Huang [4] proved the following result on complete tripartite graphs:

**Theorem 5.** *For any* $n \in \mathbb{N}$, $K_{1,1,n} \in \mathfrak{N}$ *if and only if* $n$ *is even.*

Petrosyan presented the following conjecture in the "Cycles and Colorings 2012" workshop:

**Conjecture 1.** *For any* $m, n \in \mathbb{N}$, $K_{1,m,n} \in \mathfrak{N}$ *if and only if* $\gcd(m+1, n+1) = 1$.

In this note we prove this conjecture, which also generalizes Theorem 5.

## 2. Main result

We denote the bipartition of $K_{m,n}$ by $(U, V)$, where $U = \{u_0, u_1, \dots, u_{m-1}\}$ and $V = \{v_0, v_1, \dots, v_{n-1}\}$. The interval edge-coloring $\alpha_{m,n}$ of $K_{m,n}$ with a maximum number of colors is given as follows:

$$\alpha_{m,n}(u_i v_j) = i + j + 1, \quad \text{where } 0 \le i \le m-1, 0 \le j \le n-1.$$

$K_{1,m,n}$ is a complete tripartite graph that can be viewed as a $K_{m,n}$ plus one additional vertex connected to all other vertices. In this paper we prove that if $m+1$ and $n+1$ are coprime, then it is possible to extend the $\alpha_{m,n}$ coloring of $K_{m,n}$ to an interval edge-coloring of $K_{1,m,n}$. Then we prove that if $\gcd(m+1, n+1) > 1$, then $K_{1,m,n}$ is not interval colorable.

Spectrums of the vertices for $\alpha_{m,n}$ coloring are shown in Fig. 1.

$$S(u_i, \alpha_{m,n}) = [i+1, i+n], \quad 0 \le i \le m-1$$

$$S(v_j, \alpha_{m,n}) = [j+1, j+m], \quad 0 \le j \le n-1.$$

We construct $K_{1,m,n}$ by adding a new vertex $w$ to $K_{m,n}$ and joining it with all the remaining vertices.

$$V(K_{1,m,n}) = V(K_{m,n}) \cup \{w\}$$

$$E(K_{1,m,n}) = E(K_{m,n}) \cup \{u_i w \mid 0 \le i \le m-1\} \cup \{v_j w \mid 0 \le j \le n-1\}.$$

**Theorem 6.** *If* $\gcd(m+1, n+1) = 1$, *then* $K_{1,m,n}$ *has an interval* $(m+n)$*-coloring.*

**Proof.** We color the edges $u_i v_j$ of $K_{1,m,n}$ the same way as in $\alpha_{m,n}$. In order to prove the theorem it is sufficient to show that it is possible to color the remaining edges in such a way that the following conditions are met:
(1) spectrums of vertices $u_i$ and $v_j$ remain intervals of integers
(2) spectrum of the vertex $w$ is also an interval of integers

We construct an auxiliary bipartite graph $H$ which has a bipartition $(B, C)$, where $B$ corresponds to the edges $u_i w$ and $v_j w$, and $C$ corresponds to the colors that will be used to color those edges.

$$B = \left\{u_i' \mid 0 \le i \le m-1\right\} \cup \left\{v_i' \mid 0 \le i \le n-1\right\},$$

where $u_i'$ and $v_j'$ correspond, respectively, to $u_i w$ and $v_j w$ in $E(K_{1,m,n})$.

$$C = \{c_k \mid 1 \le k \le m+n\}$$

where $c_k$ corresponds to the color $k$. We will join the vertices $b \in B$ and $c_k \in C$ if and only if we allow the edge corresponding to $b$ to receive the color $k$. Note that $|B| = |C| = m + n$.

$S(u_0, \alpha_{m,n}) = [1, n]$, so in order to satisfy the condition (1) the edge $u_0 w$ can only receive the color $n+1$ (we do not want to allow color 0). Similarly, $v_0 w$ can only be colored by $m+1$. For $u_i w$ $(1 \le i \le m-1)$ we have two options: either $i$