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Forcing faces in plane bipartite graphs (II)

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ABSTRACT

The concept of forcing faces of a plane bipartite graph was first introduced in Che and Chen (2008) [3] [Z. Che, Z. Chen, Forcing faces in plane bipartite graphs, Discrete Mathematics 308 (2008) 2427–2439], which is a natural generalization of the concept of forcing hexagons of a hexagonal system introduced in Che and Chen (2006) [2] [Z. Che and Z. Chen, Forcing hexagons in hexagonal systems, MATCH Commun. Math. Comput. Chem. 56 (2006) 649–668]. In this paper, we further extend this concept from finite faces to all faces (including the infinite face) as follows: A face s (finite or infinite) of a 2-connected plane bipartite graph G is called a *forcing face* if the subgraph $G - V(s)$ obtained by removing all vertices of s together with their incident edges has exactly one perfect matching.

For a plane elementary bipartite graph G with more than two vertices, we give three necessary and sufficient conditions for G to have all faces forcing. We also give a new necessary and sufficient condition for a finite face of G to be forcing in terms of bridges in the Z-transformation graph $Z(G)$ of G. Moreover, for the graphs G whose faces are all forcing, we obtain a characterization of forcing edges in G by using the notion of *handle*, from which a simple counting formula for the number of forcing edges follows.

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1. Introduction

An edge of a connected graph G is called a *forcing edge* if it is contained in exactly one perfect matching of G. The notion of forcing edge first appeared in a 1991 paper [6] on polyhexes by Harary et al. The root of these concepts can be traced to the works [8,11] by Randić and Klein in 1985–1987. Since then, forcing edges of perfect matchings have been investigated intensively for hexagonal systems (also called polyhexes) because they are closely related to the study of molecule resonance structures in chemistry. Most known results on forcing edges have been surveyed in [4], where some open questions and conjectures are also included.

In 1995, Zhang and Li [15] gave characterizations for a hexagonal system with forcing edges, by using the concept of Z-transformation graph of a hexagonal system introduced by Zhang et al. in [14]. In order to extend various studies on hexagonal systems, Zhang and Zhang [17] conducted an extensive study on plane elementary bipartite graphs so that many important known results for hexagonal systems can be treated in a unified way for plane bipartite graphs. In particular, they extended the concept of forcing edges from hexagonal systems to plane bipartite graphs. Motivated by their work, we introduced the concept of forcing hexagons of a hexagonal system in [2], and further generalized the above concept to forcing faces of a plane bipartite graph in [3]. Some known results on forcing hexagons and forcing faces will be presented in the next section.

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In this paper, we further extend the concept of forcing faces of a plane bipartite graph from finite faces to all faces (including the infinite face). For a plane elementary bipartite graph G with more than two vertices, we show that the following statements are equivalent:

(i) each finite face of G is forcing,
(ii) each face of G is forcing,
(iii) each perfect matching of G contains a forcing edge,
(iv) the subgraph $G - V(s_1) - V(s_2)$ has no perfect matchings for any two vertex disjoint faces s_1 and s_2 (one can be the infinite face) of G.

Moreover, we show that a finite face s of G is forcing if and only if its Z-transformation graph $Z(G)$ has exactly one bridge M_1M_2 such that the symmetric difference of M_1 and M_2 is the boundary of s.

Finally, we further study the forcing edges in a plane elementary bipartite graph G which has all faces forcing. By using the notion of handle, we obtain a characterization of forcing edges in G, from which a simple counting formula for the number of forcing edges follows.

2. Preliminaries

All graphs considered in this paper are finite, simple and connected. A *perfect matching* (or, 1-*factor*) of a graph G is a set of disjoint edges that covers all vertices of G. We assume that all graphs G in this paper have a perfect matching unless it is specified. An edge of G is called *allowed* if it is contained in a perfect matching of G, and *forbidden* otherwise. A graph G is called *elementary* if the union of all perfect matchings of G forms a connected subgraph. It is clear that any elementary graph with more than two vertices has no pendant edges and that such a graph has at least two perfect matchings. In particular, a connected bipartite graph is elementary if and only if each edge of the graph is allowed, see [9].

Let M be a perfect matching of a graph G. A cycle C of G is called an *M-alternating cycle* if its edges are alternately in M and $E(G) - M$, and we simply call C an *alternating cycle* if there is no need to specify the perfect matching M. The *symmetric difference* of two perfect matchings M and N of G, denoted by $M \oplus N$, is the set of edges contained in either M or N, but not in both. An (M, N)-*alternating cycle* of G is a cycle whose edges are in M and N alternately. It is well known [9] that the symmetric difference of two perfect matchings M and N of G is a disjoint union of (M, N)-alternating cycles. By definition, we can see that M is the unique perfect matching of G if and only if G has no M-alternating cycles. Kotzig [7] showed that if a connected graph G has a unique perfect matching M, then G has a bridge in M. Therefore, any 2-connected graph with a perfect matching has at least two perfect matchings.

A graph is called a *plane graph* if it is drawn in the plane in such a way that any two edges do not intersect, except at a common end vertex if any. A *planar embedding* of a graph G is a plane graph G' isomorphic to G. A graph is called a *planar graph* if there is a planar embedding of the graph. A plane graph divides the plane into regions which are called faces. Each bounded region is called a *finite face*, and the unbounded region is called the *infinite face*. A face s (finite or infinite) of a plane graph G is said to be *M-resonant* if the boundary of s is an M-alternating cycle with respect to some perfect matching M of G. An M-resonant face is briefly said to be *resonant* when there is no need to specify the perfect matching M. The following characterization of a plane elementary bipartite graph in terms of resonant faces was given by Zhang and Zhang (2000).

Theorem 2.1 (*[17]*). *Let G be a connected plane bipartite graph with more than two vertices. Then each face of G (including the infinite face) is resonant if and only if G is elementary.*

The concept of forcing face was first introduced in [3, Definition 1.1] as follows: In a connected plane bipartite graph G with minimum degree >1, a finite face s of G is called a *forcing face* if the subgraph $G - V(s)$ obtained by removing all vertices of s together with their incident edges has exactly one perfect matching. It is known [17] that for any connected plane bipartite graph G with minimum degree >1, if G has a forcing edge, then G is elementary. In [3] we proved the following result.

Theorem 2.2 (*Theorem 3.1 in [3]*). *For any connected plane bipartite graph G with no pendent edges, if G has a forcing (finite) face, then G is elementary.*

Here we must point out that the "finite face s" in the definition of "forcing face" should be clarified as "finite face s whose boundary is an even cycle". Otherwise, that definition would not be so meaningful. It is because the number of vertices on the boundary of a finite face s may be an odd number if the boundary of s is not a cycle, which may occur in the uninteresting case that the graph G itself has no perfect matchings although $G - V(s)$ has exactly one perfect matching. For example in Fig. 1, $G_1 - V(s)$ is an edge and so has exactly one perfect matching, but obviously G_1 has no perfect matchings. Moreover, the proof given in [3] for Theorem 2.2 implicitly used the assumption that "a forcing face is a finite face whose boundary is an even cycle". This occurred when we claimed in the proof that a forcing (finite) face of G must be in some elementary component of G. If we allow a finite face s whose boundary is not an even cycle to be included in the definition for a forcing face, then the claim mentioned above is not always valid for the graphs G in concern, and so Theorem 2.2 will not hold in general even if the graph G does have perfect matchings. This can be seen from the graph G_2 depicted in Fig. 1. It is clear that G_2 has perfect matchings, and that $G_2 - V(s)$ has exactly one perfect matching. But G_2 is not elementary since its edge e is

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