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Computing transitive closure of bipolar weighted digraphs

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ABSTRACT

We define a bipolar weighted digraph as a weighted digraph together with the sign function on the arcs such that the weight of each arc lies between 0 and 1, and no two parallel arcs have the same sign. Bipolar weighted digraphs are utilized to model so-called fuzzy cognitive maps, which are used in science, engineering, and the social sciences to represent the causal structure of a body of knowledge. It has been noted in the literature that a transitive closure of a bipolar weighted digraph contains useful new information for the fuzzy cognitive map it models.

In this paper we ask two questions: what is a sensible and useful definition of transitive closure of a bipolar weighted digraph, and how do we compute it? We give two answers to each of these questions, that is, we present two distinct models. First, we give a review of the fuzzy digraph model, which has been, in a different form and less rigorously, studied previously in the fuzzy systems literature. Second, we carefully develop a probabilistic model, which is related to the notion of network reliability.

This paper is intended for a mathematical audience.

1. Introduction

Fuzzy cognitive maps (*FCMs*) are used in science, engineering, and the social sciences to represent the causal structure of a body of knowledge (be it empirical knowledge, traditional knowledge, or a personal view); see [1,4,7,8,14–17,19–22,24] for some examples. An FCM of the type that we shall consider in this paper is described by a set of *factors* and causal relationships between pairs of factors. We call these relationships *direct impacts*. A factor can have a *direct positive* or *direct negative impact* (or both) on another factor or on itself. In addition, a numerical weight is assigned to each direct impact; these weights are usually taken to be in the interval $[0, 1]$. Fig. 1 shows a simple example of an FCM.

In this paper, we use graph-theoretic tools to analyze FCMs. In particular, we present algorithms for computing a transitive closure of the FCM, from which all, not just direct, impacts together with their weights can be read. We propose two models: in the *probabilistic model*, the absolute value of the weight of an impact is interpreted as the probability that the impact occurs, while in the *fuzzy model*, it is interpreted as the degree of truth. In both cases, the FCM is represented as a bipolar weighted directed graph; the definition of the transitive closure, however, depends on the model. In Sections 3 and 4 we shall describe several algorithms for computing the transitive closure of the bipolar weighted digraph in the fuzzy and probabilistic models, respectively, and in Section 5 we shall compare the two models, explain how to interpret the results, and summarize our contribution to the study of transitive closure of bipolar weighted digraphs.

This paper is written for a mathematical audience and (as far as we can tell) is the first to treat the subject with proper mathematical rigor and precision.

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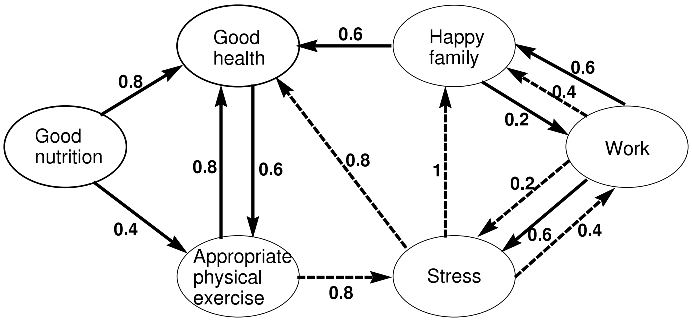

Fig. 1. A simple example of a fuzzy cognitive map (FCM). Solid arrows and dashed arrows represent positive and negative direct impacts, respectively.

2. Preliminaries

In this section we introduce relevant graph-theoretic terminology and essential tools.

A *directed graph* (shortly *digraph*) D is a triple (V, A, ψ), where V and A are disjoint sets, V is non-empty, and $\psi : A \to V \times V$ is a function. The sets $V = V(D)$ and $A = A(D)$ are called the *vertex set* and the *arc set* of D, and ψ is called the *incidence function* of D. If $\psi(a) = (u, v)$, then vertices u and v are called the *tail* and *head* of the arc a, respectively. Arcs $a, a' \in A$ are called *parallel* if $\psi(a) = \psi(a')$. An arc $a \in A$ with $\psi(a) = (u, u)$ for some $u \in V$ is called a *loop*. In general, we shall allow both parallel arcs and loops in our digraphs. When no confusion can arise (in particular, for bipolar digraphs, as defined below), we shall write shortly $D = (V, A)$ and $a = (u, v)$ instead of $D = (V, A, \psi)$ and $\psi(a) = (u, v)$.

If u is a vertex in a digraph D, then $\mathrm{indeg}_D(u)$ and $\mathrm{outdeg}_D(u)$ denote the number of arcs in D with head u and the number of arcs in D with tail u, respectively.

Throughout this paper, the set $\{-1, 1\}$ will be denoted by Ω.

Definition 2.1. A *bipolar digraph* is a digraph $D = (V, A, \psi)$, where $A \subseteq V^2 \times \Omega$ and $\psi(u, v, \sigma) = (u, v)$ for all $u, v \in V$ and $\sigma \in \Omega$. We write shortly $D = (V, A)$ instead of $D = (V, A, \psi)$, and define the *sign* of an arc $a = (u, v, \sigma)$ as $\mathrm{sign}(a) = \sigma$ for all $a \in A$.

A *bipolar weighted digraph* $D = (V, A, w)$ is a bipolar digraph (V, A) together with a weight function $w : A \to [0, 1]$.

If a bipolar weighted digraph is used to model an FCM, then its vertices represent the factors, and the arcs represent the direct impacts of the FCM. In particular, the arcs of negative and positive sign represent the direct negative and direct positive impacts, respectively.

A *directed* (u, v)*-walk* in a digraph $D = (V, A, \psi)$ is a sequence $W = u_0a_1u_1a_2u_2 \dots a_ku_k$ of vertices and arcs of D such that $u = u_0$, $v = u_k$; $u_0, u_1, \dots, u_k \in V$; $a_1, \dots, a_k \in A$, and $\psi(a_i) = (u_{i-1}, u_i)$ for all $i \in \{1, 2, \dots, k\}$. We shall assume a directed walk W contains at least one arc so that its *length* k is at least 1. The directed walk W is *closed* if $u_0 = u_k$. A *directed path* in D is a directed walk with all vertices pairwise distinct, except possibly the initial and terminal vertex; in the latter case, the directed path is called a *directed cycle*. For a directed walk $W = u_0a_1u_1a_2u_2 \dots a_ku_k$ in a bipolar digraph we define the *sign* of W as $\mathrm{sign}(W) = \mathrm{sign}(a_1) \dots \mathrm{sign}(a_k)$. If W' is a directed (u, v)-walk and W'' is a directed (v, z)-walk, then the concatenation of W' and W'' is a directed (u, z)-walk denoted by $W'W''$. Clearly, $\mathrm{sign}(W'W'') = \mathrm{sign}(W')\mathrm{sign}(W'')$.

A few words about what the above definitions mean for an FCM. When a bipolar weighted digraph D is used to model an FCM, a factor u is said to *impact* factor v if there is a directed (u, v)-walk W in D. This impact is said to be *positive* if $\mathrm{sign}(W) = 1$ and *negative* if $\mathrm{sign}(W) = -1$.

In the FCM literature, what we call direct impacts are interpreted in (at least) two different ways: as implications (e.g. [11]) or as causal inferences (e.g. [10]). In the first case, an arc $a = (u, v)$ is interpreted as "u implies v" if $\mathrm{sign}(a) = 1$, and as "u implies not v" or "not u implies v" if $\mathrm{sign}(a) = -1$. In the second case, an arc $a = (u, v)$ is interpreted as "an increase in u causes an increase in v" if $\mathrm{sign}(a) = 1$, and as "an increase in u causes a decrease in v" or "a decrease in u causes an increase in v" if $\mathrm{sign}(a) = -1$. We shall not limit ourselves to any one of these two interpretations, however, we shall make two basic assumptions throughout this work. The first assumption is that an arc $a = (u, v)$ with $\mathrm{sign}(a) = -1$ should be interpreted *both* as "u implies not v" *and* "not u implies v" in the first case, and *both* as "an increase in u causes a decrease in v" *and* "a decrease in u causes an increase in v" in the second case. This assumption is essential for the definition of the sign of a directed walk to make sense.

Our second basic assumption is that the relation "impact", if we ignore the signs, is transitive, while the signs of direct impacts are combined using the first assumption. To be precise, let u, v, and z be three factors of the FCM. If u impacts v positively, and v impacts z positively, then u impacts z positively. If u impacts v negatively, and v impacts z negatively, then u impacts z positively. If u impacts v positively, and v impacts z negatively, or vice-versa, then u impacts z negatively. The two assumptions just described are the reasoning behind the definition of the sign of a directed walk.

We continue with a few more technical definitions. We shall use the symbol $\langle e_1, \dots, e_n \rangle$ to denote the *multiset* consisting of (not necessarily distinct) elements $e_1, \dots, e_n$. The *multiplicity* $\mu(e_i, M)$ of an element e_i in a multiset $M = \langle e_1, \dots, e_n \rangle$