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Further generalizations of the Wythoff game and the minimum excludant

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ABSTRACT

Given non-negative integers a and b, we consider the following game $WYT(a, b)$. Given two piles that consist of x and y matches, and two players having alternate turns; a single move consists of a player choosing x' matches from one pile and y' from the other such that

$$0 \le x' \le x, 0 \le y' \le y, 0 < x' + y', \quad \text{and} \quad [\min(x', y') < b \text{ or } |x' - y'| < a].$$

The player who takes the last match is the winner in the normal version of the game and the loser in its misère version.

It is easy to verify that the cases $(a = 0, b = 1)$, $(a = b = 1)$, and $(b = 1, \forall a)$ correspond to the two-pile, Wythoff and Fraenkel NIM, respectively. The concept of the minimum excludant, *mex*, is known to be instrumental in solving the last two games. We generalize this concept by introducing a function mex_b (such that $mex_1 = mex$) to solve the normal and misère versions of the game $WYT(a, b)$.

1. The game $WYT(a, b)$ and its special cases

The game $WYT(a, b)$ was defined in the abstract. By this definition, a player can take any number of matches from one pile and at most $b - 1$ from the other (that is, $\min(x', y') < b$), or he can take two amounts that differ by at most $a - 1$ (that is, $|x' - y'| < a$), yet, in both cases he is not allowed to pass his turn (that is, $x' + y' > 0$).

If $a = 0$, $b = 1$, we get the standard (and trivial) NIM with two piles. Indeed, the second option, $|x' - y'| < a$, becomes impossible and, hence, either $x' = 0$ or $y' = 0$, but not both.

If $a = b = 1$, then a player can take either

(i) any positive number of matches from one pile and none from the other (that is, $x' + y' > 0$ and $\min(x', y') = 0$), or
(ii) the same positive number of matches from each pile (that is, $x' = y' > 0$).

Thus, $WYT(1, 1)$ coincide with the classical game introduced in 1907 by Wythoff [18].

In [8,9], Fraenkel generalized this game, replacing the equality $x' = y'$ in (ii) by a weaker constraint $|x' - y'| < a$. The resulting game is $WYT(a, 1)$. In this paper, we also replace the equality $\min(x', y') = 0$ in (i) with a weaker constraint $\min(x', y') < b$, getting $WYT(a, b)$.

Remark 1. We could generalize even further by replacing the inequality $\min(x', y') < b$ with $((x' < b \, \forall y')$ or $(y' < c \, \forall x'))$. Yet, in Section 7, we will see that the resulting game $WYT(a, b, c)$ is trivial unless $b = c$. In that section, we will consider two more simple cases, $a = 0$ and $b = 0$, but in Sections 1–6, we assume that $a > 0$ and $b > 0$.

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Table 1
The first ten P-positions of $WYT(a, b)$.

$(a=b=1)$			$(a=2, b=1)$		
n	x_n	y_n	n	x_n	y_n
0	0	0	0	0	0
1	1	2	1	1	3
2	3	5	2	2	6
3	4	7	3	4	10
4	6	10	4	5	13
5	8	13	5	7	17
6	9	15	6	8	20
7	11	18	7	9	23
8	12	20	8	11	27
9	14	23	9	12	30

The positions of $WYT(a, b)$ are the pairs (x, y), where x and y denote the numbers of matches in the two piles. By default, we will assume that $x \le y$.

Furthermore, (x, y) is called a *P-position* if the player who enters it (the Previous player) can win. Otherwise, (x, y) is called an *N-position*, since in this case, the player who leaves it (the Next player) can win. Clearly, each move from a P-position leads to an N-position and for every N-position, there is a move to a P-position.

Remark 2. To solve a game separately, it is sufficient to find all its N- or P-positions. To solve it for enabling play in a sum, its Sprague–Grundy function [17,12] has to be computed. Yet, this is difficult already for the standard Wythoff game; see, for example, [1,2,16]. For this reason, we will not consider sums in this paper.

Due to the symmetry of $WYT(a, b)$, a pair (x, y) is a P-position if and only if (y, x) is a P-position.

Obviously, there is a unique terminal position $(0, 0)$, since $b > 0$. By definition, $(0, 0)$ is a P-position in the normal version of $WYT(a, b)$ and an N-position in its misère version.

In this paper, both the normal and misère versions are recursively solved, namely, we obtain a recursive formula for the P-positions.

2. The solution of Fraenkel's game

Let us start with $b = 1$. In this case, the game $WYT(a, 1) = WYT(a)$ was solved by Fraenkel; see [8,9] for the standard and misère versions, respectively (and also [10,11] for some related games). We postpone the discussion of the misère version until Section 6, where it will be considered in the more general setting of $WYT(a, b)$. As for the standard version of $WYT(a)$, the set of its P-positions $\{(x_n, y_n) \mid n = 0, 1, \ldots\}$ was characterized in [8] by the following recursion:

$$x_n = mex\{x_i, y_i \mid 0 \le i < n\}, \qquad y_n = x_n + an, \quad n \ge 0, \tag{1}$$

where the *minimum excludant* function $mex(S)$ is defined for any subset $S \subset \mathbb{Z}_+$ of the non-negative integers as the minimum $z \in \mathbb{Z}_+$ such that $z \notin S$; in particular, $mex(\emptyset) = 0$.

The first ten P-positions of the games $WYT(1)$ and $WYT(2)$ are given in Table 1.

Moreover, Fraenkel solved the recursion and got the following explicit formula for (x_n, y_n).

Let $\alpha = \alpha(a) = \frac{1}{2}(2 - a + \sqrt{a^2+4})$ be the (unique) positive root of the quadratic equation $\frac{1}{z} + \frac{1}{z+a} = 1$. In particular, we have $\alpha(1) = \frac{1}{2}(1 + \sqrt{5})$, which is the *golden section (or ratio)*, and $\alpha(2) = \sqrt{2}$. Then,

$$x_n = \lfloor \alpha n \rfloor, \qquad y_n = x_n + an \equiv \lfloor n(\alpha + a) \rfloor; \quad n \ge 0. \tag{2}$$

As mentioned in [8], the explicit formula (2) solves the game $WYT(a)$ in linear time, in contrast to recursion (1) providing only an exponential algorithm.

3. The recursive solution of $WYT(a, b)$ based on mex_b

The function *mex* can be generalized as follows. Given a finite subset $S \subseteq \mathbb{Z}_+$ of m non-negative integers. Let us order S and extend it by $s_{m+1} = \infty$ and by $s_0 = -b$, to get a sequence $s_0 < s_1 < \cdots < s_m < s_{m+1}$. Obviously, there is a unique minimum $i \in \{0, 1, \ldots, m\}$ such that $s_{i+1} - s_i > b$. By definition, we set $mex_b(S) = s_i + b$.

It is easily seen that the function mex_b is well-defined, $mex_b(\emptyset) = 0$ and $mex_1 = mex$.

We will show that the recursion (1) can be naturally extended to the game $WYT(a, b)$.

Theorem 1. *The set of P-positions* $\{(x_n, y_n) \mid n = 0, 1, \ldots\}$ *of the game* $WYT(a, b)$ *is determined by the same recursive formula* (1)*, in which mex is replaced with* mex_b*, namely:*

$$x_n = mex_b\{x_i, y_i \mid 0 \le i < n\}, \qquad y_n = x_n + an; \quad n \ge 0. \tag{3}$$

The first ten P-positions of the games $WYT(1, 2)$ and $WYT(2, 3)$ are given in Table 2.