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Recursive method to solve the problem of "Gambling with God"

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ABSTRACT

Suppose Alice gambles with God who is the dealer. There are n total rounds in the game and God can choose any m rounds to win and the other $n-m$ rounds to lose. At first Alice has holdings a. In each round, Alice can increase her holdings by q times the amount she wagers if she wins. So what strategy should Alice take to ensure the maximum total holdings in the end? And how much is the total final holdings? It is called the "Gambling with God" problem. In this paper, a recursive method is proposed to solve the problem, which shows the extensive application of recursive methods.

1. Introduction

Suppose Alice gambles with God who is the dealer. There are n total rounds in the game and God can choose any m rounds to win and the other $n-m$ rounds to lose. God's goal is to minimize Alice's winnings. Alice has holdings a at the beginning. In each round, Alice can increase her holdings by q times the amount she wagers if she wins. So what strategy should Alice take to ensure the maximum total holdings when the game is over? And how much is the total final holdings? This problem is called the "Gambling with God" problem. It was first proposed in an old American newspaper in a simpler case of $q=1$, $a=100$, $n=10$ and $m=1$. The newspaper's answer was 9309. Unfortunately, it was not easy to find the newspaper again. In this paper we generalize the original problem to the above version with general q, n and m.

2. Main result

In order to understand the problem quickly, we consider the simplest case of $n=1$ and $m=0$. Alice gambles with God for only one round and will definitely win. Then Alice should wager all her holdings a. Because the odds are q to 1, Alice will gain $(q+1)a$ as the final holdings. In general, if $m=0$, for any $n\geq 1$, Alice will gamble with God for n rounds and never lose. Then Alice will wager all she has in each round. So her final holdings would be $(q+1)^n a$.

Consider the case of $m=1$, which means that God can choose any one round to win. We have the following theorem.

Theorem 1. *When $m=1$, if Alice gambles with God for n rounds, the best way for Alice to win as much as she can is to wager $\frac{(n-1)q}{nq+1}a$ in the first round, and the final holdings would be $\frac{(q+1)^n}{nq+1}a$.*

Proof. We prove it by induction.

First we consider the simplest case of $n=1$. Alice only gambles one round with God and knows she would lose it definitely. Then she will wager $0=\frac{(1-1)q}{q+1}a$. Her final holdings would be $a=\frac{(q+1)^1}{q+1}a$. The result is true.

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Now assume that the assertion is true when $n = k$. Then consider the case of $n = k + 1$. Alice will gamble $k + 1$ rounds with God and God will win only one round. Suppose Alice wagers x in the first round. There will be two possibilities:

1. Alice wins the first round. Then Alice has the holdings $a + qx$. There are k rounds later and God will only win one round. So Alice will gain the final holdings $\frac{(q+1)^k}{kq+1}(a + qx)$ according to the induction hypothesis.
2. Alice loses the first round. Then Alice has the holdings $a - x$. There are k rounds later and God will lose each round. So Alice will wager all she has for each round and gain the final holdings $(q + 1)^k(a - x)$.

In order to gain the biggest holdings, the following equality should hold,

$$\frac{(q+1)^k}{kq+1}(a+qx) = (q+1)^k(a-x).$$

Solving this equation, we get $x = \frac{kq}{(k+1)q+1}a$. The final holdings will be

$$(q+1)^k(a-x) = \frac{(q+1)^{k+1}}{(k+1)q+1}a.$$

The result is true, which, from the principle of induction, proves the theorem. □

For the case of $m = 2$, the result will be more complex.

Theorem 2. *When $m = 2$, if Alice gambles with God for $n \geq 2$ rounds, the best way for Alice to win as much as she can is to wager $\frac{\binom{n-1}{2}q}{\binom{n}{2}q^2+nq+1}a$ in the first round. The final holdings would be $\frac{(q+1)^n}{\binom{n}{2}q^2+nq+1}a$.*

Proof. We will argue by induction too.

This result is trivial for $n = 2$. Assume the assertion is true when $n = k$. Then consider the case of $n = k + 1$. Alice will gamble $k + 1$ rounds with God and God will win only two rounds. Suppose Alice wagers x in the first round. There are two possibilities:

1. Alice wins the first round. Then Alice has the holdings $a + qx$. There are k rounds later and God will win two of them. So Alice will gain the final holdings $\frac{(q+1)^k}{\binom{k}{2}q^2+kq+1}(a + qx)$ according to the induction hypothesis.
2. Alice loses the first round. Then Alice has the holdings $a - x$. There are k rounds later and God will win only one round. According to Theorem 1, Alice will gain the final holdings $\frac{(q+1)^k}{kq+1}(a - x)$.

In order to gain the biggest final holdings, the following equality should hold,

$$\frac{(q+1)^k}{\binom{k}{2}q^2+kq+1}(a+qx) = \frac{(q+1)^k}{kq+1}(a-x).$$

Solving this equation, we get

$$x = \frac{\binom{k}{2}q^2}{\binom{k+1}{2}q^2+(k+1)q+1}a.$$

The final holdings will be

$$\frac{(q+1)^k}{kq+1}(a-x) = \frac{(q+1)^{k+1}}{\binom{k+1}{2}q^2+(k+1)q+1}a.$$

The result is true, which completes the proof. □

For the general m, the problem becomes complicated. To simplify the discussion, we first denote some symbols. Suppose Alice will gamble with God for n rounds and God can choose any m rounds to win and the other $n - m$ rounds to lose. Let $f(n, m), n \geq m \geq 0$ represent the ratio between the holdings Alice will wager in the first round and her original holdings before the gambling. Let $S(n, m), n \geq m \geq 0$ represent the ratio between the final holdings of Alice and her original holdings. It is easy to see that

$$f(n,0) = 1, \qquad S(n,0) = (q+1)^n.$$