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On α-total domination in graphs

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ABSTRACT

Let $G = (V, E)$ be a graph with no isolated vertex. A subset of vertices S is a total dominating set if every vertex of G is adjacent to some vertex of S. For some α with $0 < \alpha \leq 1$, a total dominating set S in G is an α-total dominating set if for every vertex $v \in V \setminus S$, $|N(v) \cap S| \geq \alpha|N(v)|$. The minimum cardinality of an α-total dominating set of G is called the α-total domination number of G. In this paper, we study α-total domination in graphs. We obtain several results and bounds for the α-total domination number of a graph G.
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1. Introduction

In this paper, we continue the study of total domination in graphs which is well studied in graph theory. Let $G = (V, E)$ be a graph with vertex set V, edge set E and no isolated vertex. A *dominating set* D in a graph G is a set of vertices of G such that each vertex not in D is adjacent to a vertex of D, while a *total dominating set*, denoted TDS, of G is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. A TDS of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [7,6]. A recent survey of total domination in graphs can be found in [9].

For notation and graph theory terminology we in general follow [7]. Specifically, let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E\}$ and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. The degree of v is $d_G(v) = |N_G(v)|$. If the graph G is clear from the context, then we simply write $N(v)$, $N[v]$ and $d(v)$ rather than $N_G(v)$, $N_G[v]$ and $d_G(v)$, respectively. The minimum and maximum degree among the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. The order of G is given by $n(G) = |V(G)|$ and its size by $m(G) = |E(G)|$. A cycle on n vertices is denoted by C_n, while a path on n vertices is denoted by P_n. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. We denote the set of leaves of G by $L(G)$. For a subset $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. A *packing* in G is a set of vertices that are pairwise at distance at least 3 apart in G.

A vertex and an edge are said to *cover* each other in a graph G if they are incident in G. A *vertex cover* in G is a set of vertices that covers all the edges of G, while a *total vertex cover* in G, abbreviated TVC, is a vertex cover that induces a subgraph with no isolated vertex. The minimum cardinality among all the TVCs in G is called the *total vertex covering number* of G and is denoted by $\mathrm{tvc}(G)$. A TVC in G of cardinality $\mathrm{tvc}(G)$ is called a $\mathrm{tvc}(G)$-cover.

A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a *maximum matching*. The number of edges in a maximum matching of G is called the *matching number* of G which we denote by $\alpha'(G)$.

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In this paper we study a variant of total domination in graphs, called α-total domination. For some α with $0 < \alpha \leq 1$, we say that a TDS S in G is an *α-total dominating set*, abbreviated by αTDS, if for every vertex $v \in V \setminus S$, $|N(v) \cap S| \geq \alpha|N(v)|$. Thus, every vertex v outside the TDS S has at least $\alpha|N(v)|$ neighbors inside S. The minimum cardinality of an α-TDS of G is called the α-total domination number of G and is denoted by $\gamma_{\alpha t}(G)$. An αTDS of G of cardinality $\gamma_{\alpha t}(G)$ is called a $\gamma_{\alpha t}(G)$-set. Every graph without isolated vertices has a αTDS, since $S = V$ is such a set.

The analogous concept of α-domination in graphs was introduced by Dunbar et al. [4] who defined a dominating set D to be an *α-dominating set*, abbreviated by αDS, if for every vertex $v \in V \setminus D$, we have $|N(v) \cap D| \geq \alpha|N(v)|$. The minimum cardinality of an αDS of G is called the α-domination number of G and is denoted by $\gamma_\alpha(G)$. An αDS of G of cardinality $\gamma_\alpha(G)$ is called a $\gamma_\alpha(G)$-set. The concept of α-domination in graphs has been studied, for example, in [3,5] and elsewhere.

2. Properties of α-total dominating sets

In this section, we present basic properties of αTDSs in graphs. Throughout this section, let $G = (V, E)$ be a graph with no isolated vertex and with maximum degree $\Delta = \Delta(G)$, and let α satisfy $0 < \alpha \leq 1$.

Since every αTDS is a TDS, we observe that $\gamma_t(G) \leq \gamma_{\alpha t}(G)$ for all α. Further for $0 < \alpha \leq 1/\Delta$ and for every vertex $v \in V$, we observe that $\alpha|N(v)| \leq 1$, and so in this case an αTDS S in G simply requires that every vertex outside S is adjacent to at least one vertex inside S. Thus every TDS is an αTDS, whence $\gamma_{\alpha t}(G) = \gamma_t(G)$. Therefore α-total domination in graphs is a generalization of total domination in graphs.

Let α satisfy $(\Delta - 1)/\Delta < \alpha \leq 1$. We observe that in this range, every αTDS is a TVC in the graph. To see this, let S be an arbitrary αTDS in G. For each vertex $v \in V \setminus S$, we have that

$$|N(v) \cap S| \geq \alpha|N(v)| > \left(\frac{\Delta - 1}{\Delta}\right)|N(v)| = \left(1 - \frac{1}{\Delta}\right)d(v) \geq d(v) - 1,$$

implying that $|N(v) \cap S| \geq d(v)$. Hence, $N(v) \subseteq S$ for every vertex $v \in V \setminus S$, and so $V \setminus S$ is an independent set in G. Equivalently, the set S is a total vertex cover in G. Since S is an arbitrary αTDS in G, we have that $\mathrm{tvc}(G) \leq \gamma_{\alpha t}(G)$. On the other hand, every total vertex cover in G is an αTDS, and so $\gamma_{\alpha t}(G) \leq \mathrm{tvc}(G)$. Consequently, $\gamma_{\alpha t}(G) = \mathrm{tvc}(G)$ for $(\Delta - 1)/\Delta < \alpha \leq 1$. Therefore α-total domination in graphs is a generalization of total vertex cover in graphs.

Since every αTDS is an αDS, we observe that $\gamma_\alpha(G) \leq \gamma_{\alpha t}(G)$ for all α with $0 < \alpha \leq 1$. If there exists a $\gamma_\alpha(G)$-set S such that $G[S]$ has no isolated vertex, then S is an αTDS, implying that $\gamma_{\alpha t}(G) \leq |S| = \gamma_\alpha(G)$, and so $\gamma_\alpha(G) = \gamma_{\alpha t}(G)$. Let D be a $\gamma_\alpha(G)$-set. For each vertex $v \in D$, let v' denote an arbitrary neighbor of v in G and let $D' = \cup_{v \in D}\{v'\}$. Then the set $D \cup D'$ is an αTDS, and so $\gamma_{\alpha t}(G) \leq |D \cup D'| \leq 2|D| = 2\gamma_\alpha(G)$. Further if D is not a packing in G, then we can choose D' so that $|D \cup D'| < 2|D|$.

If M is a maximum matching in G and D is the set consisting of the $2|M|$ vertices of G incident with edges in M, then D is an αTDS, implying that $\gamma_{\alpha t}(G) \leq 2\alpha'(G)$ for all α.

Our earlier remarks, together with the definition of an αTDS, readily imply the following observation, which summarizes fundamental properties of α-total dominating sets in a graph.

Observation 1. *Let G be a graph of order n with no isolated vertex and with maximum degree Δ. Let α satisfy $0 < \alpha \leq 1$. Then the following holds.*

(a) $\max\{\gamma_t(G), \gamma_\alpha(G)\} \leq \gamma_{\alpha t}(G) \leq \min\{n, 2\gamma_\alpha(G), 2\alpha'(G), \mathrm{tvc}(G)\}$.
(b) *For $0 < \alpha \leq 1/\Delta$, we have $\gamma_{\alpha t}(G) = \gamma_t(G)$.*
(c) *For $(\Delta - 1)/\Delta < \alpha \leq 1$, we have $\gamma_{\alpha t}(G) = \mathrm{tvc}(G)$.*
(d) *If $0 < \alpha_1 \leq \alpha_2 \leq 1$, then $\gamma_{\alpha_1 t}(G) \leq \gamma_{\alpha_2 t}(G)$.*
(e) *$\gamma_{\alpha t}(G) = 2\gamma_\alpha(G)$ if and only if every $\gamma_\alpha(G)$-set is a packing in G.*
(f) *$\gamma_{\alpha t}(G) = \gamma_\alpha(G)$ if and only if there is a $\gamma_\alpha(G)$-set S such that $G[S]$ has no isolated vertex.*

3. Exact values

In this section, we determine exact values of the α-total domination number for special classes of graphs. It is known (see [4]) that for a complete graph K_n, $\gamma_\alpha(K_n) = \lceil\alpha(n-1)\rceil$ for all α with $0 < \alpha \leq 1$. Hence by Observation 1(e), we have the result of Proposition 2. However for completeness, we provide a short proof of this result.

Proposition 2. *If K_n is a complete graph with $n \geq 2$ vertices, then for all α with $0 < \alpha \leq 1$, we have $\gamma_{\alpha t}(K_n) =$* $\max\{2, \lceil\alpha(n-1)\rceil\}$.

Proof. Let $G = K_n$. By Observation 1, $\gamma_{\alpha t}(G) \geq \gamma_t(G) = 2$. Hence we may assume that $\lceil\alpha(n-1)\rceil > 2$, for otherwise the desired bound is immediate. If D is an $\gamma_{\alpha t}(G)$-set, then for every vertex $v \in V \setminus D$, we have $|N(v) \cap D| \geq \lceil\alpha(n-1)\rceil$, and so $\gamma_{\alpha t}(G) = |D| \geq \lceil\alpha(n-1)\rceil$. To prove the reverse inequality, let S be an arbitrary subset of vertices in G such that $|S| = \lceil\alpha(n-1)\rceil$. By assumption, $|S| > 2$ and so S is a TDS in G. For every vertex $v \in V \setminus S$, we have $|N(v) \cap S| = |S| = \lceil\alpha(n-1)\rceil = \lceil\alpha|N(v)|\rceil \geq \alpha|N(v)|$, implying that S is an αTDS. Hence, $\gamma_{\alpha t}(G) \leq |S| = \lceil\alpha(n-1)\rceil$. Consequently, $\gamma_{\alpha t}(G) = \lceil\alpha(n-1)\rceil$. □

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