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Note

A note on entire choosability of plane graphs

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ABSTRACT

A plane graph is called entirely k-choosable if for any list assignment L such that $|L(x)| = k$ for each $x \in V(G) \cup E(G) \cup F(G)$, we can assign each element x a color from its list such that any two elements that are adjacent or incident receive distinct colors. Wang and Lih (2008) [5] conjectured that every plane graph is entirely $(\Delta + 4)$-choosable and showed that the conjecture is true if $\Delta \geq 12$. In this note, we prove that (1) Every plane graph G with $\Delta \geq 7$ is entirely $(\Delta + 4)$-choosable. (2) Every plane graph G with $\Delta \geq 6$ is entirely $(\Delta + 5)$-choosable.

1. Introduction

Graphs considered in this note are finite, simple and undirected. Unless stated otherwise, we follow the notations and terminology in [1].

For a plane graph G, we denote its vertex set, edge set, face set, and minimum degree by $V(G)$, $E(G)$, $F(G)$ and $\delta(v)$, respectively. For a vertex v, $d_G(v)$ and $N_G(v)$ denote its degree and the set of its neighbors in G, respectively.

We use $b(f)$ to denote the boundary walk of a face f and write $f = [v_1v_2v_3 \cdots v_n]$ if $v_1, v_2, v3, \ldots, v_n$ are the vertices of $b(f)$ in cyclic order. The *degree*, $d(f)$, of a face f is the number of edges in its boundary $b(f)$, cut edges being counted twice. A *k-vertex* (or *k-face*) is a vertex (or a face) of degree k, a *k^--vertex* (or *k^--face*) is a vertex (or a face) of degree at most k, and a *k^+-vertex* (or *k^+-face*) is defined similarly.

Two faces of a plane graph are said to be *adjacent* if they have at least one common boundary edge. For $x \in V(G) \cup F(G)$, we use $F_k(x)$ and $V_k(x)$ to denote the set of all k-faces and k-vertices that are incident or adjacent to x, respectively. For $f \in F(G)$, we write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are on the boundary of f in clockwise order.

A *k-coloring* of G is a mapping ϕ from $V(G)$ to a set of size k such that $\phi(x) \neq \phi(y)$ for any adjacent vertices x and y. A graph is k-colorable if it has a k-coloring.

A list-assignment L to the vertices of G is an assignment of a set $L(v)$ of colors to vertex v for every $v \in V(G)$. If G has a coloring ϕ such that $\phi(v) \in L(v)$ for all vertices v, then we say that G is *L-colorable* or ϕ is an L-coloring of G. We say that G is *k-list colorable* (or k-choosable) if it is L-colorable for every list-assignment L satisfying $|L(v)| = k$ for all vertices v.

An *entire coloring* of a plane graph G is a coloring of the faces, vertices, and edges of G, which we call the elements of G, so that all incident or adjacent elements receive distinct colors; an entire list-coloring is defined analogously. In 1972, Kronk and Mitchem [2] conjectured that any plane graph of maximum degree Δ is entirely $(\Delta + 4)$-colorable and proved this conjecture for $\Delta = 3$ [3]. In [4], it is proved that the conjecture is true if $\Delta \geq 6$. More recently, Wang and Zhu [6] completely settled the conjecture. For the list version of entire coloring, Wang and Lih [5] conjectured that every plane graph is entirely $(\Delta + 4)$-choosable. They proved this conjecture for $\Delta \geq 12$ and also proved that every plane graph with $\Delta \geq 9$ is entirely $(\Delta + 5)$-choosable.

In this note, we prove the following results.

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Theorem 1.1. *Every plane graph with maximum degree $\Delta \geq 7$ is entirely $(\Delta + 4)$-choosable.*

Theorem 1.2. *Every plane graph with maximum degree $\Delta \geq 6$ is entirely $(\Delta + 5)$-choosable.*

For convenience, we introduce the following terminology. A *partial* (*entire*) *coloring* is an entire coloring, except that some elements may not be colored. Given a partial coloring of G, a color α is forbidden to an element $x \in V(G) \cup E(G) \cup F(G)$ if α appears on another element y which is adjacent or incident with x. Let $f = [uvw]$ be a 3-face of G with $d(u) \geq d(v) \geq d(w)$. If $d(w) = 4$, then f is called *a bad* 3-*face* and uw and vw are called special edges. We use $F_3(v)$ and $n_3(v)$ to denote the set of incident 3-faces and the number of adjacent 3-vertices of a vertex v, respectively.

2. Proof of Theorem 1.1

In this section, a 5-vertex v is called *bad* if $|F_3(v)| = 5$. Moreover, we use $F_3'(v)$ to denote the set of bad 3-faces incident with v. Similarly, we use $F_3''(v)$ to denote the set of incident 3-faces of a vertex v such that every 3-face in $F_3''(v)$ is incident with a bad 5-vertex.

We will prove Theorem 1.1 by contradiction. Hence, we suppose that G is a counterexample to Theorem 1.1 with $\sigma(G) = |E(G)| + |V(G)|$ as minimal as possible. That is, there exists a list assignment L with $|L(x)| = \Delta + 4$ for all $x \in V(G) \cup E(G) \cup F(G)$ such that G is not entirely $(\Delta + 4)$-choosable.

We first prove some structural lemmas about the minimal counterexample.

Lemma 2.1. $\delta(G) \geq 3$.

Proof. Assume that G contains a 2^--vertex v. If $d(v) = 1$, assume u is the neighbor of v. By the minimality of G, $G' = G - v$ admits an entirely $(\Delta + 4)$-coloring ϕ from its lists, which is also a partial coloring of G. Note that v and uv are uncolored. By simply counting, at most $\Delta + 1$ colors are forbidden to uv and three colors are forbidden to v. We can easily extend ϕ of G' to G.

Now, assume that $d(v) = 2$ and u, w be the neighbors of v. Let f_1 and f_2 be the two faces incident with v. If both $d(f_1) \geq 5$ and $d(f_2) \geq 5$, we contract uv to u and obtain G'. By the choice of G, G' admits an entire $(\Delta + 4)$-coloring ϕ from its lists which induces a partial entire coloring of G with v and uv uncolored. Note that at most $\Delta + 3$ colors are forbidden to uv, we first properly color uv. Since at most six colors are forbidden to v, v can receive a proper color. Therefore, by symmetry, we assume that $d(f_1) \leq 4$. By the choice of G, $G' = G - uv$ is entirely $(\Delta + 4)$-choosable, except that f_1 and uv uncolored. We erase the color on v, then sequentially assign $uv f_1$ and v proper colors from its lists and obtain an entire list coloring of G. Therefore, $\delta(G) \geq 3$. □

Lemma 2.2. *If $f = [uvw]$ is a* 3-*face of G, then* $\min\{d(u), d(v), d(w)\} \geq 4$.

Proof. Suppose that $d(v) = 3$ and $N(v) = \{x, u, w\}$. Consider $G - vw$. $G - vw$ admits an entire coloring using $(\Delta + 4)$ colors. First we erase the color assigned on v and we obtain a partial entire list coloring of G with f, vw and v uncolored. Note that at most $\Delta + 3$ colors are forbidden to vw, we can properly color vw. Then we give proper colors to v and f in sequence to extend this partial coloring to the whole graph. □

Lemma 2.3. *Let $f = [uvw]$ be a* 3-*face of G with $d(u) \geq d(v) \geq d(w)$. If $d(w) = 4$, then $d(u) = d(v) = \Delta$.*

Proof. Suppose that $d(u) \leq \Delta - 1$. By the minimality of G, $G - wu$ is entirely $(\Delta + 4)$-choosable. Let ϕ be such a coloring of $G - wu$ from its lists, ϕ is a partial entire coloring of G with wu, f uncolored. We erase the color on w and properly color wu. Note that under ϕ, at most $\Delta + 3$ colors are forbidden to uw, thus the above is possible. Then we properly color w and f sequentially. Hence, ϕ can be extended to the whole graph. □

Lemma 2.4. *Let $f = [uvwx]$ be a* 4-*face of G. If $d(x) = 3$, then $d(u) = d(w) = \Delta$.*

Proof. W.l.o.g, suppose that $d(u) \leq \Delta - 1$. Let f_1 be the adjacent face of f sharing the common edge xu. By the choice of G, $G - ux$ admits an entire coloring ϕ using $(\Delta + 4)$ colors, which is a partial coloring of G with f, xu uncolored. Let ϕ_1 be the coloring induced from ϕ by erasing the color assigned on x. We will extend ϕ_1 to the whole graph as follows. First we properly color f note that at most 10 colors are forbidden to f. Then we color xu and x in sequence. By simply counting, at most $\Delta + 3$ colors are forbidden to xu and nine colors are forbidden to x. □

Lemma 2.5. *G contains no two adjacent bad* 3-*faces sharing a special edge.*

Proof. Suppose that $f_1 = [xyu]$ and $f_2 = [xyv]$ are two bad 3-faces sharing the special edge xy. By definition, assume that $d(x) = 4$. Consider $G - xy$. Assume $f = xvyu$ be the new face by deleting xy. By the choice of G, $G - xy$ admits an entire coloring ϕ from its list using $(\Delta + 4)$ colors. To extend ϕ to the whole graph G, we first erase the color assigned on x and f to obtain a partial coloring of G with xy, x, f_1 and f_2 uncolored. Then we properly color xy, x, f_1 and f_2 sequentially. With a similar discussion on the above lemma, we can obtain an entire list coloring of G, a contradiction. □