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Note

On graphs for which the connected domination number is at most the total domination number

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ABSTRACT

In this note, we give a finite forbidden subgraph characterization of the connected graphs for which any non-trivial connected induced subgraph has the property that the connected domination number is at most the total domination number. This question is motivated by the fact that any connected dominating set of size at least 2 is in particular a total dominating set. It turns out that in this characterization, the total domination number can equivalently be substituted by the upper total domination number, the paired-domination number and the upper paired-domination number, respectively. Another equivalent condition is given in terms of structural domination.

A *dominating set* of a graph G is a vertex subset such that every vertex of G belongs to X or has a neighbor in X. The minimum size of a dominating set of G, the *domination number*, is denoted by $\gamma(G)$. A *total dominating set* X of G is a vertex subset such that every vertex of G has a neighbor in X. That is, X is a dominating set and the subgraph induced by X, henceforth denoted by $G[X]$, does not have an isolated vertex. Note that any graph that does not have an isolated vertex has a total dominating set (and vice versa). The minimum size of a total dominating set of G is denoted by $\gamma_t(G)$ and is called the *total domination number* of G. A total dominating set of minimum size is called a *minimum total dominating set*. The maximum size of an inclusion wise minimal total dominating set, the *upper total domination number*, is denoted by $\Gamma_t(G)$. Total domination has been introduced by Cockayne et al. [4] and is well-studied now. A survey of some recent results is given by Henning [8]. A variant of (total) domination is paired-domination. A *paired-dominating set* of G is a dominating set X such that $G[X]$ has a perfect matching. In particular, any paired-dominating set is a total dominating set. Furthermore, paired-dominating sets always exist in graphs that do not have isolated vertices. The minimum size of a paired-dominating set is denoted by $\gamma_p(G)$ and is called the *paired-domination number* of G. Similar to the total domination case one defines the *upper paired-domination number* $\Gamma_p(G)$. Apparently, paired-domination was first studied by Haynes and Slater [7].

Another variant of domination is connected domination. A *connected dominating set* of G is a dominating set such that $G[X]$ is connected. Clearly, a graph has a connected dominating set iff it is connected. The minimum size of a connected dominating set, the *connected domination number*, is denoted by $\gamma_c(G)$.

One can say that total domination and connected domination (together with independent domination) belong to the most intensively studied variants of domination. There are a lot of sharp bounds on γ_t and γ_c and for many graph classes we know the computational complexity of the two parameters. Although a little less studied yet, similar things can be mentioned about paired-domination. Still a good introduction into the theory of domination is given by the book of Haynes et al. [6]. The property that two parameters are equal for all induced subgraphs is usually called *perfection* of the two parameters. Finding the forbidden induced subgraph characterization for a certain type of perfection, in particular for parameters from

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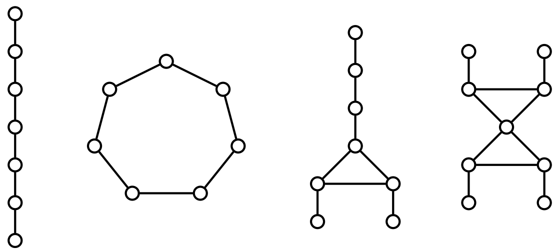

Fig. 1. The graphs P_7, C_7, F_1 and F_2.

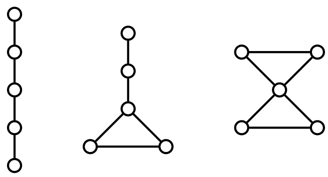

Fig. 2. The graphs P_5, G_1 and G_2.

the context of domination, seems to be accepted as a step in the understanding of the relation of the parameters involved. A prominent example for the perfection of two domination parameters are the so-called domination perfect graphs. A graph is *domination perfect* iff for any induced subgraph the domination number equals the minimum size of an independent dominating set. After the problem was open for some time, a forbidden induced subgraph characterization of the domination perfect graphs was finally given by Zverovich and Zverovich [11]. A characterization of the connected graphs for which in any connected subgraph $\gamma = \gamma_c$ holds is given by Zverovich [10]. An extension of this result to total domination and clique-domination was given by Goddard and Henning [5]. We call a connected graph *non-trivial* if it is not an isolated vertex. It is clear that any connected dominating set of size at least 2 is also a total dominating set. Thus any connected graph with $\gamma_c \geq 2$ fulfills $\gamma_c \geq \gamma_t$. However, an open problem seems to be the characterization of the connected graphs for which we can find, in any non-trivial connected induced subgraph, a minimum total dominating set that is connected, i.e. $\gamma_c \leq \gamma_t$. These graphs then fulfill $\gamma_c = \gamma_t$, provided $\gamma_c \geq 2$. Graphs for which the connected domination number equals the total domination number were studied before by Chen [3], but he only studies trees and unicyclic graphs with this property.

The following theorem gives a characterization of the connected graphs for which any non-trivial connected induced subgraph fulfills $\gamma_c \leq \gamma_t$, in terms of forbidden induced subgraphs. Somewhat surprisingly, it turns out that in this characterization γ_t can be substituted by any of the parameters Γ_t, γ_p and Γ_p. Furthermore, the set of forbidden induced subgraphs yields the equivalence of another condition in terms of structural domination.

Theorem 1. *Let G be a connected graph. The following conditions are equivalent:*

1. *Any non-trivial connected induced subgraph of G fulfills $\gamma_c \leq \gamma_t$.*
2. *Any non-trivial connected induced subgraph of G fulfills $\gamma_c \leq \Gamma_t$.*
3. *Any non-trivial connected induced subgraph of G fulfills $\gamma_c \leq \gamma_p$.*
4. *Any non-trivial connected induced subgraph of G fulfills $\gamma_c \leq \Gamma_p$.*
5. *G is $\{P_7, C_7, F_1, F_2\}$-free (see Fig. 1).*
6. *Any connected induced subgraph H of G has a connected dominating set X such that $H[X]$ is $\{P_5, G_1, G_2\}$-free (see Fig. 2).*

We observe that the class of connected $\{P_7, C_7, F_1, F_2\}$-free graphs properly contains the class of connected split graphs. It is well-known that the computation of the domination number γ in split graphs is *NP*-complete [2]. From [5], it follows that in any non-trivial connected $\{P_5, C_5\}$-free graph γ equals γ_c and γ_t, provided $\gamma \geq 2$. Thus, the computation of the parameters γ_c and γ_t remains *NP*-complete if the instances are restricted to split graphs. Therefore, computing the parameters γ_c and γ_t on connected $\{P_7, C_7, F_1, F_2\}$-free graphs remains *NP*-complete.

In view of the forbidden subgraphs of Theorem 1 (see Figs. 1 and 2) we obtain the following immediate consequence.

Corollary 1. *Let G be a $\{C_3, C_7\}$-free graph. The following statements are equivalent.*

1. *Any non-trivial connected induced subgraph fulfills $\gamma_c \leq \gamma_t$ ($\gamma_c \leq \Gamma_t$, $\gamma_c \leq \gamma_p$, $\gamma_c \leq \Gamma_p$ respectively).*
2. *G is P_7-free.*
3. *Any connected induced subgraph H of G has a connected dominating set X such that $H[X]$ is P_5-free.*

Note that any bipartite graph is in particular $\{C_3, C_7\}$-free. Hence, Corollary 1 applies to bipartite graphs.

The main step of the proof of Theorem 1 is formulated in the following lemma.