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Uniform-cost inverse absolute and vertex center location problems with edge length variations on trees

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ABSTRACT

This article considers the inverse absolute and the inverse vertex 1-center location problems with uniform cost coefficients on a tree network T with $n+1$ vertices. The aim is to change (increase or reduce) the edge lengths at minimum total cost with respect to given modification bounds such that a prespecified vertex s becomes an absolute (or a vertex) 1-center under the new edge lengths. First an $O(n \log n)$ time method for solving the height balancing problem with uniform costs is described. In this problem the height of two given rooted trees is equalized by decreasing the height of one tree and increasing the height of the second rooted tree at minimum cost. Using this result a combinatorial $O(n \log n)$ time algorithm is designed for the uniform-cost inverse absolute 1-center location problem on tree T. Finally, the uniform-cost inverse vertex 1-center location problem on T is investigated. It is shown that the problem can be solved in $O(n \log n)$ time if all modified edge lengths remain positive. Dropping this condition, the general model can be solved in $O(r_v n \log n)$ time where the parameter r_v is bounded by $\lceil n/2 \rceil$. This corrects an earlier result of Yang and Zhang.

1. Introduction

Inverse location problems have become an important aspect of optimization in recent years due to their role in practice and theory. Whereas classical location problems deal with finding the optimal location of one or more new facilities on network systems or in the space in order to satisfy the demands of customers optimally (see e.g. [8,9,14,15]), the goal of inverse location optimization is to modify specific parameters (like edge lengths or vertex weights in a network, point weights or point coordinates on space) of a given location problem at minimum total cost within certain modification bounds such that a given feasible solution of the problems becomes an optimal solution with respect to the new parameter values.

One of the well-known models in inverse location optimization is the inverse version of the center problems. The earliest study in this direction was performed by Cai et al. [7] in 1999 in order to prove the $\mathcal{NP}$-hardness of the inverse vertex center problem with uniform cost coefficients on directed graphs. Later, Gassner [12] showed that the inverse unit-weight k-centrum problem with uniform cost coefficients on a tree can be solved in $O(n^3k^2)$ time. This implies an $O(n^3)$ time algorithm for the uniform-cost inverse absolute center location problem on trees. In 2008, Yang and Zhang [17] developed an $O(n^2 \log n)$ time solution method for the inverse vertex 1-center location problem on trees which is based on linear programming arguments. Furthermore, the authors proposed an $O(n)$ time approach for solving the uniform-cost case. We show in this article by a counter example that in general this solution approach does not work correctly. The inverse 1-center location problem with edge length augmentation was treated by Alizadeh et al. [2]. Using a sequence of self-defined AVL-

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search trees, they designed an exact combinatorial algorithm with time complexity of $O(n \log n)$ on tree networks. Moreover, it was shown that the problem can be solved in $O(n)$ time if all cost coefficients are uniform. Recently, Alizadeh and Burkard [1] investigated the inverse absolute (and vertex) 1-center location problems on trees with arbitrary nonnegative cost for increasing or reducing the edge lengths. They developed a combinatorial $O(n^2)$ time algorithm for the inverse absolute 1-center location problem in which no topology change occurs on the given tree. Dropping this condition, an exact $O(n^2 r)$ time algorithm was proposed for the general model where $r, r < n$, is the compressed depth of the underlying tree. Moreover, the authors showed that the inverse *vertex* 1-center location problem can be solved by a new approach with improved $O(n^2)$ time complexity, if all modified edge lengths remain positive. Finally, it was also shown that in the general case one gets an improved $O(n^2 r_v)$ time complexity where the parameter r_v is bounded by $\lceil n/2 \rceil$.

Concerning inverse median problems see [5,6,11,12,4,10,3,16].

In this article we consider the uniform-cost inverse absolute and vertex 1-center location problems with edge length variations on a tree network. We develop new solution methods with improved time complexities. The remainder of this article is outlined as follows: In the next section, we describe the underlying models. Then we recall some fundamental properties from the classical absolute (and vertex) center location problems in order to derive the essential solution ideas for solving the problems under investigation. In Section 3.1, an $O(n)$ time greedy algorithm is stated for solving the uniform-cost tree height reduction problem. The construction of a corresponding height-reduction cost function of a rooted tree is treated in Section 3.2. This function returns the minimum cost for reducing the height of the given tree by any feasible amount. The results of the Subsections are applied in Section 4 in order to balance the heights of two rooted trees optimally in $O(n \log n)$ time by increasing the height of one tree and reducing the height of the other one at minimum total (uniform) cost. A new combinatorial algorithm with improved $O(n \log n)$ time complexity for the uniform-cost inverse absolute 1-center location problem on trees is developed in Section 5. Finally, in Section 6, the uniform-cost inverse vertex 1-center location problem on a tree is solved in $O(n \log n)$ time, provided that the modified edge lengths remain positive. Dropping this condition, one can get an $O(r_v n \log n)$ time solution method for the general case where $r_v \le \lceil n/2 \rceil$.

2. Problem definition and basic properties

Let an undirected tree $T = (V(T), E(T))$ with vertex set $V(T)$, $|V(T)| = n + 1$, and edge set $E(T)$ be given. Every edge $e \in E(T)$ has a positive length $\ell(e)$. We say that a point p lies in T, $p \in T$, if p coincides with a vertex or lies on an edge of T. In the *classical absolute (or vertex)* 1-*center location problem* on the given tree T, the goal is to find a point $p \in T$ (or $p \in V(T)$ respectively,) such that the maximum distance from any vertex $v \in V(T)$ to point p becomes a minimum. Define

$$f_\ell(p) = \max_{v \in V(T)} d_\ell(v, p)$$

where $d_\ell(v, p)$ denotes the shortest path distance from v to p with respect to edge lengths ℓ on T. Then the absolute (or vertex) 1-center problem can be stated as

$$\begin{aligned} &\text{minimize} \quad f_\ell(p) \\ &\text{subject to} \quad p \in T \, (\text{or } p \in V(T)). \end{aligned} \tag{1}$$

A point p^* which solves problem (1), is said to be an *absolute* 1-*center* (or a *vertex* 1-*center*, respectively). In 1973, Handler [13] proposed linear solution algorithms for determining the absolute and the vertex 1-center of a tree network.

The *uniform-cost inverse absolute (or vertex)* 1-*center location problem with edge length variations on trees* can be stated as follows:

Let s be a prespecified vertex on the given tree network T. We want to modify the edge lengths ℓ to $\tilde{\ell}$ at minimum total cost so that the prespecified vertex s becomes an absolute (or a vertex) 1-center of tree T. Every edge length $\ell(e)$ can only be modified between a lower bound $\ell_{\text{low}}(e) \ge 0$ and a upper bound $\ell_{\text{upp}}(e)$. Moreover, we assume that the cost for modifying each length $\ell(e)$ by one unit is the same, say 1. Therefore, the total cost is measured by the following linear function

$$\sum_{e \in E(T)} (x(e) + y(e)),$$

where $x(e)$ is the amount by which the length $\ell(e)$ is increased and $y(e)$ is the amount by which the length $\ell(e)$ is reduced. A solution vector (x, y) with $x = \{x(e) : e \in E(T)\}$ and $y = \{y(e) : e \in E(T)\}$ is called feasible if it guarantees that the vertex s is an absolute (or a vertex) 1-center of T and all bounds for the edge lengths are met. Thus the uniform-cost inverse absolute (or vertex) 1-center location problem on T can be written as the following *nonlinear semi-infinite (or nonlinear) optimization model*:

$$\begin{aligned} &\text{minimize} \quad \sum_{e \in E(T)} (x(e) + y(e)) \\ &\text{subject to} \quad f_{\tilde{\ell}}(s) \le f_{\tilde{\ell}}(p) && \text{for all } p \in T \, (\text{or } p \in V(T)), \\ &\qquad \tilde{\ell}(e) = \ell(e) + x(e) - y(e) && \text{for all } e \in E(T), \\ &\qquad 0 \le x(e) \le \ell^+(e) && \text{for all } e \in E(T), \\ &\qquad 0 \le y(e) \le \ell^-(e) && \text{for all } e \in E(T), \end{aligned} \tag{2}$$

where we set $\ell^+(e) = \ell_{\text{upp}}(e) - \ell(e)$ and $\ell^-(e) = \ell(e) - \ell_{\text{low}}(e)$.