Discrete Applied Mathematics 159 (2011) 717–726

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Separator orders in interval, cocomparability, and AT-free graphs

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ARTICLE INFO

Article history:
Received 12 January 2010
Received in revised form 2 January 2011
Accepted 13 January 2011
Available online 11 February 2011

Keywords:
Graph
Minimal separator
Partial order

ABSTRACT

We introduce a new type of order of sets of vertices. Using this concept, we describe the structure and the relationship between chordal, interval, cocomparability, and asteriodal triple-free graphs.

1. Introduction

Chordal graphs, interval graphs, cocomparability graphs, and asteriodal triple-free (AT-free) graphs are well studied. One reason is that they have a wide range of applications (see Section 8.4 of [16] or Sections 2.4 and 3.4 of [24] for some examples). Another reason is that their structural properties admit efficient algorithms to many problems that are NP-hard in general (e.g. finding a maximum cardinality independent set takes polynomial time on chordal [13], interval, cocomparability [16], and AT-free [5] graphs). In this paper, we characterise these graph classes in terms of their minimal separators. Our characterisations make the relationships shown in Fig. 1.1 immediate.

These characterisations are phrased in terms of separator orders, which we now explain through a geometric analogy. The sides of a line are the two connected components (open half-planes) that result from removing that line from the plane. Analogously, the sides of a set of vertices are the connected components that result from removing that set of vertices from a connected graph. For example, the sides of $\{v\}$ in Fig. 1.2 are $\{t, u\}$, $\{w, x\}$, and $\{y, z\}$. Two lines are parallel if one line intersects at most one side of the other. We define the parallel relation between sets of vertices the same way. Chordal graphs are characterised by this notion.

Theorem 1 (*Parra and Scheffler [27]*)**.** *A graph is chordal if and only if every two minimal separators are parallel.*

The parallel relation between lines is transitive, symmetric, and reflexive. Each equivalence class has a total order $\preceq$ such that $L \preceq J \preceq K$ implies that L and K do not intersect the same side of J. Given a set of lines, we can obtain a partial order by ordering each equivalence class independently. The resulting partial order has two important properties: (a) two elements are comparable if and only if they are parallel and (b) if $L \preceq J \preceq K$, then J separates $L \setminus J$ from $K \setminus J$. We call any partial order satisfying these two conditions a *separator order*.

The parallel relation between sets of vertices is different than the parallel relation between lines. First, the parallel relation is not transitive in general. Second, even when transitivity holds, we cannot always construct a separator order: in Fig. 1.2, $\{\{u\}, \{v\}, \{w\}\}$ has a separator order whereas $\{\{u\}, \{w\}, \{y\}\}$ does not. In this paper, we explore when we can create separator orders. Specifically, we prove three main results.

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0166-218X/$ – see front matter
doi:10.1016/j.dam.2011.01.014

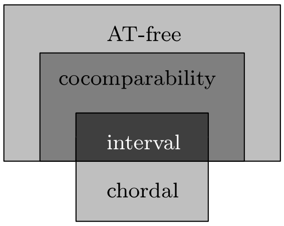

Fig. 1.1. Venn diagram of various graph classes.

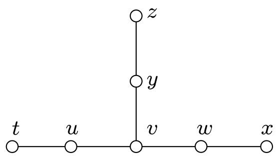

Fig. 1.2. The minimal separators of a tree are the internal vertices.

(1) A graph is interval if and only if the family of its minimal separators has a total separator order.
(2) A graph is cocomparability if and only if the family of its minimal separators has a partial separator order.
(3) A graph is AT-free if and only if every pairwise parallel family of minimal separators has a total separator order.

We close this paper by explaining why we suspect that the last characterisation can be strengthened.

2. Related work

An *intersection model* for a graph is a mapping of vertices to sets such that two vertices are adjacent if their corresponding sets have a non-empty intersection. A graph is chordal if it can be represented as the intersection of subtrees of a tree [6,14,29]. Similarly, a graph is interval if it can be represented as the intersection of subpaths of a path [11]. In [24], this intersection perspective is used to prove many of the relationships illustrated in Fig. 1.1.

If one minimises the size of the tree (path) from which the model of a chordal (interval) graph is chosen, each edge of the tree (path) corresponds to a minimal separator [17,23]. Intuitively, this correspondence explains the separator characterisations of chordal and interval graphs described earlier: every two separators in a chordal graph are parallel because one edge lies on one side of another in a tree; minimal separators in an interval graph are totally ordered because the edges of a path are totally ordered. Although, these intuitions are only approximately correct, our proofs follow along these lines.

A cocomparability order is a transitive orientation of the complement of a graph (defined in the next section). Sometimes it is defined as a linear extension of such an orientation. An interval order is a cocomparability order of an interval graph (this definition differs from others such as [16], but it is equivalent by the results of [15]). In this paper, we show that a graph has a cocomparability (interval) order if and only if the set of minimal separators has a partial (total) separator order. One important difference between a cocomparability order and a separator order is in the "size" of what they order. To be more concrete, let n be the number of vertices in a graph. A star graph has $\Theta(n^2)$ non-edges but only one minimal separator. Alternatively, we can construct a cocomparability graph with $\Theta(n^2)$ non-edges and $\Theta(2^{n/2})$ minimal separators (consider two cliques, where each vertex in one clique is adjacent to exactly one vertex in the other).

As an aside, although a cocomparability graph may have an exponential number of minimal separators in terms of n, interval graphs have at most $O(n)$ minimal separators (this follows from correspondence mentioned above between minimal separators in the interval graph and edges in its path intersection model). The proofs of our characterisations are straightforward constructions. Moreover, the minimal separators of any given graph can be listed in time polynomial in the number of separators in the graph [19]. For interval graphs, a total separator order can be constructed in time polynomial in n. For cocomparability graphs, a partial separator order can be constructed in time polynomial in the number of minimal separators in the graph.

Interval graphs are characterised in terms of consecutive 1's orders [11]. Separator orders are a special type of consecutive 1's order. In Section 5, we explore the connection between these two concepts.

Our notion of a separator order generalises the previous notion of a cut-set lattice [10]. Every cut-set lattice is a separator order, but not vice versa.

3. Preliminaries

We restrict our attention to simple (i.e. no loops or multiple edges), undirected, finite graphs. Let $G = (V, E)$ denote a graph G with vertex set V and edge set E. Let $uv \in E$ signify that vertices u and v are adjacent in G. A *clique* is a set of pairwise adjacent vertices. An *independent set* is a set of pairwise non-adjacent vertices.