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Rainbow domination on trees

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ABSTRACT

This paper studies a variation of domination in graphs called rainbow domination. For a positive integer k, a k-rainbow dominating function of a graph G is a function f from $V(G)$ to the set of all subsets of $\{1, 2, \ldots, k\}$ such that for any vertex v with $f(v) = \emptyset$ we have $\cup_{u \in N_G(v)} f(u) = \{1, 2, \ldots, k\}$. The 1-rainbow domination is the same as the ordinary domination. The k-rainbow domination problem is to determine the k-rainbow domination number $\gamma_{rk}(G)$ of a graph G, that is the minimum value of $\sum_{v \in V(G)} |f(v)|$ where f runs over all k-rainbow dominating functions of G. In this paper, we prove that the k-rainbow domination problem is NP-complete even when restricted to chordal graphs or bipartite graphs. We then give a linear-time algorithm for the k-rainbow domination problem on trees. For a given tree T, we also determine the smallest k such that $\gamma_{rk}(T) = |V(T)|$.

1. Introduction

Domination and its variations in graphs are natural models for the location problems in operations research. They have been extensively studied in the literature; see [4,7,8]. A *dominating set* of a graph G is a subset D of $V(G)$ such that every vertex not in D is adjacent to some vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The following variation of domination was introduced by Brešar, Henning and Rall [2].

For a positive integer k, we use $[k]$ to denote the set $\{1, 2, \ldots, k\}$, and $2^{[k]}$ the set of all subsets of $[k]$. A *k-rainbow dominating function* of G is a function $f : V(G) \to 2^{[k]}$ such that for every vertex v, either $f(v) \neq \emptyset$ or $f(N_G(v)) = [k]$, where $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $f(S) = \cup_{x \in S} f(x)$ for any subset S of $V(G)$. The *weight* of f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The *$k$-rainbow domination number* $\gamma_{rk}(G)$ of G is the minimum weight of a k-rainbow dominating function. A k-rainbow dominating function f of G is *optimal* if $w(f) = \gamma_{rk}(G)$. The *k-rainbow domination problem* is to determine the k-rainbow domination number of a given graph. Notice that the ordinary domination is the same as the 1-rainbow domination if we view a dominating set D as a 1-rainbow dominating function f defined by $f(v) = \{1\}$ when $v \in D$ and $f(v) = \emptyset$ otherwise.

The *Cartesian product* of two graphs G and H is the graph $G \Box H$ with vertex set $V(G \Box H) = V(G) \times V(H)$ and edge set $E(G \Box H) = \{(u, v)(u', v') : u = u' \text{ with } vv' \in E(H) \text{ or } uu' \in E(G) \text{ with } v = v'\}$. Rainbow domination of a graph G coincides with the ordinary domination of the Cartesian product of G with the complete graph, that is $\gamma_{rk}(G) = \gamma(G \Box K_k)$ (see [2]).

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Hartnell and Rall [6] established several results on rainbow domination. In particular, it was proved in [6] that

$$\min\{|V(G)|, \gamma(G) + k - 2\} \leq \gamma_{rk}(G) \leq k\gamma(G)$$

for any $k \geq 2$ and any graph G. Their attempt to characterize graphs with $\gamma(G) = \gamma_{r2}(G)$ was inspired by the following famous conjecture by Vizing [9].

Vizing's Conjecture. For any graph G and H, we have $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.

One of the related problems posted by Hartnell and Rall [5] is to find classes of graphs that achieve the equality. They showed that $\gamma(G\Box H) = \gamma(G)\gamma(H)$ if G is a graph with $\gamma(G) = \gamma_{r2}(G)$ and H is a so-called *generalized comb*.

Brešar, Henning and Rall [2] introduced rainbow domination to study the relation with paired-domination; also see [1]. They gave a linear-time algorithm for finding a minimum weighted 2-rainbow dominating function of a tree. On the other hand, Brešar and Šumenjak [3] proved that the 2-rainbow domination problem is NP-complete even when restricted to chordal graphs or bipartite graphs. They also established exact values of the 2-rainbow domination numbers for paths, cycles and suns, and upper and lower bounds for the generalized Petersen graphs.

The purpose of this paper is to study k-rainbow domination for a general k. In Section 2, we prove that the k-rainbow domination problem is NP-complete even when restricted to chordal graphs or bipartite graphs, and then give a linear-time algorithm for the k-rainbow domination problem on trees. For a graph G on n vertices, $\gamma_{ri}(G) \leq \gamma_{ri+1}(G) \leq n$ for any i and $\gamma_{rn}(G) = n$. In Section 3, we determine the minimum k such that $\gamma_{rk}(T) = |V(T)|$ for any tree T.

2. Complexity in *k*-rainbow domination

In this section, we prove that the k-rainbow domination problem is NP-complete even when restricted to chordal graphs or bipartite graphs. We then give a linear-time algorithm for the k-rainbow domination problem on trees.

The domination problem is known to be NP-complete not only for general graphs but also for chordal graphs and bipartite graphs and many other classes of graphs; see [4]. This is also the case for the k-rainbow domination problem.

Theorem 1. *For any positive integer k, the k-rainbow domination problem is NP-complete for general graphs.*

Proof. We shall prove the theorem by reducing the k-rainbow domination problem to the domination problem. Given a graph G on n vertices, consider the graph G' with the vertex set $V(G') = V(G) \cup \{v_2, v_3, \ldots, v_k : v \in V(G)\}$ and edge set $E(G') = E(G) \cup \{vv_i : v \in V(G), 2 \leq i \leq k\}$. Namely, we add $n(k-1)$ leaves to G by joining $k-1$ leaves to each vertex of G (we call a degree-1 vertex of G a *leaf* of G). We claim that G has a dominating set of cardinality at most s if and only if G' has a k-rainbow dominating function of weight at most $s + n(k-1)$.

Suppose G has a dominating set D of cardinality at most s. Consider the function f from $V(G')$ to $2^{[k]}$ defined by

$$f(u) = \begin{cases} \{1\}, & \text{if } u \in D, \\ \emptyset, & \text{if } u \in V(G) - D, \\ \{i\}, & \text{if } u = v_i \text{ for some } v \in V(G) \text{ and } 2 \leq i \leq k. \end{cases}$$

Suppose $f(u) = \emptyset$. By the definition of f, $u \in V(G) - D$ and so u has a neighbor $v \in D$. Since $f(v) = \{1\}$ and $f(u_i) = \{i\}$ for $2 \leq i \leq k$, we have $f(N_{G'}(u)) = [k]$. Therefore, f is a k-rainbow dominating function of G'. Also, the weight of f is $|D| + n(k-1) \leq s + n(k-1)$.

On the other hand, suppose G' has a k-rainbow dominating function f of weight at most $s + n(k-1)$. For each $v \in V(G)$, we may assume that $\sum_{2\leq i\leq k} |f(v_i)| \leq k-1$ for otherwise if $\sum_{2\leq i\leq k} |f(v_i)| \geq k$ then we replace each $f(v_i)$ by $\{i\}$ and add 1 to the set $f(v)$ to obtain a k-rainbow dominating function of weight at most $s + n(k-1)$. Now, consider the set $D = \{v \in V(G) : f(v) \neq \emptyset\}$. For any vertex $v \in V(G) - D$, we have $f(v) = \emptyset$ and so $f(N_{G'}(v)) = [k]$. As $\sum_{2\leq i\leq k} |f(v_i)| \leq k-1$, we have $f(u) \neq \emptyset$ for some $u \in N_G(v)$ which implies $u \in D$. Therefore, D is a dominating set of G. Next, we calculate the cardinality of D. Suppose there are n' vertices $v \in V(G)$ such that $f(v_i) = \emptyset$ for some i. For these n' vertices v we have $f(v) = [k]$. Therefore, the weight of f is at least $n'k + (|D| - n') + n'0 + (n - n')(k-1) = |D| + n(k-1)$ implying $|D| + n(k-1) \leq s + n(k-1)$ and so $|D| \leq s$. □

Notice that if G is chordal or bipartite, then so is G' in the proof above. We thus have

Corollary 2. *For any positive integer k, the k-rainbow domination problem is NP-complete for chordal graphs and for bipartite graphs.*

In the rest of this section, we establish a linear-time algorithm for the k-rainbow domination problem on trees. For technical reasons, we in fact dealing with a more general problem. A *k-rainbow assignment* is a mapping L that assigns each vertex v a label $L(v) = (a_v, b_v)$ with $a_v, b_v \in \{0\} \cup [k]$. A *$k$-$L$-rainbow dominating function* is a function $f : V(G) \to 2^{[k]}$ such that for every vertex v in G we have

(L1) $|f(v)| \geq a_v$, and
(L2) $|f(N_G(v))| \geq b_v$ whenever $f(v) = \emptyset$.