Available online at www.sciencedirect.com
ScienceDirect
Discrete Applied Mathematics 156 (2008) 3464–3474
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

On diagnosability of large multiprocessor networks

R. Ahlswede*, H. Aydinian

Department of Mathematics, University of Bielefeld, POB 100131, D-33501 Bielefeld, Germany

Received 7 August 2007; accepted 11 February 2008
Available online 20 March 2008

Abstract

We consider problems of fault diagnosis in multiprocessor systems. Preparata, Metze and Chien [F.P. Preparata, G. Metze, R.T. Chien, On the connection assignment problem of diagnosable systems, IEEE Trans. Comput. EC 16 (12) (1967) 848–854] introduced a graph theoretical model for system-level diagnosis, in which processors perform tests on one another via links in the system. Fault-free processors correctly identify the status of tested processors, while the faulty processors can give arbitrary test results. The goal is to identify faulty processors based on the test results. A system is said to be t-diagnosable if faulty units can be identified, provided the number of faulty units present does not exceed t. We explore here diagnosis problems for n-cube systems and give bounds for diagnosability of the n-cube. We also describe a simple diagnosis algorithm A which is linear in time and which can be used for sequential diagnosis as well as for incomplete diagnosis in one step. In particular, the algorithm applied to arbitrary topology based interconnection systems G with N processors improves previously known ones. It has sequential diagnosability $t_A(G) \geq \lceil 2N^{\frac{1}{2}} \rceil - 3$, which is optimal in the worst case.

Keywords: System-level diagnosis; PMC model; Sequential diagnosis; Diagnosability

1. Introduction

The concept of system-level diagnosis was introduced by Preparata, Metze and Chien [12] to perform automatic fault diagnosis in multiprocessor systems. In their graph theoretical model, called PMC model, a system S is composed of independent units $u_1, \ldots, u_n$ connected by communication links. The system is represented as an undirected graph $G = (V, E)$, where the vertices represent units and edges represent interconnection links. In the PMC model diagnosis is based on a suitable set of tests between units. A unit u_i can test u_j iff the vertices corresponding to u_i and u_j in the graph $G = (V, E)$ of the system S are adjacent. The outcome of a test in which u_i tests u_j is denoted by a_{ij}, where $a_{ij} = 1$ if u_i finds u_j to be faulty and $a_{ij} = 0$ if u_i finds u_j to be fault-free.

The basic conditions of the PMC model are the following:

- The fault-free units give correct test outcomes.
- The answers of faulty units are unreliable.
- The number of faulty units t is bounded and all faults are permanent.

* Corresponding author. Tel.: +49 521 106 4789; fax: +49 521 106 6481.
E-mail address: ahlswede@Mathematik.Uni-Bielefeld.DE (R. Ahlswede).

0166-218X/$ - see front matter
doi:10.1016/j.dam.2008.02.001

The set of tests for the purpose of diagnosis is represented by a set of directed edges where the presence of oriented edge (u_i, u_j) means that u_i tests u_j. Given a faulty set of units $F \subset V$ the set of all test outcomes $\{a_{ij}\}$ is called *syndrome*. The task is to identify the faulty units based on a syndrome produced by the system. In [12] two different kinds of strategies were introduced for implementing the diagnosis approach.

One-step diagnosis (or diagnosis without repair): a system is called t-fault diagnosable (or shortly t-diagnosable) in one step, if all faulty units can be uniquely identified from any syndrome, provided the number of faulty units does not exceed t.

Sequential diagnosis (or diagnosis with repair): a system is called sequentially t-diagnosable if it can identify at least one faulty unit from any syndrome, provided the number of faulty units does not exceed t. Under a sequential diagnosis strategy a system can locate a faulty unit, repair it and then repeat the process until all faulty units are repaired.

The *degree of diagnosability*, or simply diagnosability, of a system graph G is defined (for both kinds of strategies) as the maximum t such that the system is t-diagnosable.

The PMC model has been widely studied (see [4] for a good survey). It is known [12] that the maximum degree of diagnosability of a one-step diagnosis algorithm for any system is bounded from above by the minimum vertex degree of the interconnection graph. However, the real commercial multiprocessor systems are based on topologies of graphs with small average vertex degree (like grids, hypercubes, cube-connected cycles, trees etc.).

Sequential diagnosis is a much more powerful strategy than one-step t-fault diagnosis. On the other hand the sequential diagnosis has the disadvantage of repeated execution of diagnosis and repair phases and may be time consuming for large systems.

That was the motivation for developing diagnosis algorithms (see [5]) which are able to diagnose in one step the status of a large fraction of the system units (i.e. if a “large” subset F' of the actual fault set F can be identified from any syndrome, provided $|F| \le t$). This approach is referred to as *incomplete diagnosis in one step*.

Clearly, incomplete diagnosis in one-step amounts to *one-step diagnosis* if $F' = F$, while sequential diagnosis is a strategy using multiple steps of incomplete diagnosis to identify and replace all faulty units. A common and usually implicit assumption is that no additional faults are introduced while the process of sequential diagnosis is going on.

Since the degree of diagnosability is different for different kinds of diagnosis strategies the notation $t_0(G)$ for one-step diagnosis and $t(G)$ for incomplete diagnosis in one step is appropriate. Actually the paper is concerned with $t(G)$ only except for Corollary 3, which provides an upper bound to the number of iterations in sequential diagnosis. Moreover, Section 4.3 deals with the diagnosability of a general graph G under algorithm A and here we use the notation $t_A(G)$.

The *diagnostic graph* DG of a system graph $G = (V, E)$, corresponding to a given syndrome, consists of bidirectional arcs, between every two neighbors of the original graph G, labelled by 0 or 1. Let $\{u, v\} \in E(G)$, then the presence of oriented edges (u, v) and (v, u) with $a_{uv} = 1$ and $a_{vu} = 0$ implies that v is faulty. In the following we assume that a diagnostic graph does not contain such “trivial” configurations. Thus the outcomes of any two neighbors coincide. Therefore we can represent a diagnostic graph as an undirected graph where each edge is labelled by a 0 or a 1.

Given a syndrome, a subset F of the vertex set V is called a *consistent fault set* if the assumption that the vertices in F are faulty and those in $V \setminus F$ are fault-free is consistent with the syndrome. The following simple facts are useful for obtaining upper and lower bounds for the diagnosability of a system graph.

Fact 1. Given a syndrome, let $F_1, \ldots, F_k$ be a collection of consistent fault sets with $|F_i| \le t$; $i = 1, \ldots, k$. Then G is not sequentially t-diagnosable if $\bigcap_{i=1}^k F_i = \varnothing$ and $\bigcup_{i=1}^k F_i = V$.

Given a diagnostic graph DG, define the subgraph G_0 consisting of edges labelled only by 0 (0-edges). The connected components of the graph G_0 will be called 0-components of DG.

Fact 2. All vertices of a 0-component have the same status: “faulty” or “fault-free”.

This fact was used for lower estimates of diagnosabilities under sequential diagnosis (of certain regular graphs) and for the incomplete diagnosis approach in [9,5]. For a given system, let the number of faults be bounded by t. Suppose then that given a syndrome the diagnostic graph contains a 0-component K of size $|K| > t$. Then clearly all vertices