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Algorithms for media

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Abstract

Falmagne recently introduced the concept of a *medium*, a combinatorial object encompassing hyperplane arrangements, topological orderings, acyclic orientations, and many other familiar structures. We find efficient solutions for several algorithmic problems on media: finding short reset sequences, shortest paths, testing whether a medium has a closed orientation, and listing the states of a medium given a black-box description.

Keywords: Media theory; Partial cubes; Graph algorithms; Shortest paths; Reset sequences

1. Introduction

Motivated by political choice theory, Falmagne [16] (see also [17]) recently introduced the concept of a medium, a combinatorial object that also encompasses hyperplane arrangements, topological orderings, acyclic orientations, and many other familiar combinatorial structures.

Formally, a *medium* consists of a finite set of *states* transformed by the actions of a set of *tokens*, satisfying certain axioms. That is, it is essentially a restricted type of deterministic finite automaton, without distinguished initial and final states. Tokens may be concatenated to form *messages* (words, in finite automaton terminology). We use upper case letters to denote states, and lower case letters to denote tokens and messages; Sw denotes the state formed by applying the tokens in message w to state S. A token t is said to have a *reverse* $\tilde{t}$ if, for any two states $S \neq Q$, $St = Q$ iff $Q\tilde{t} = S$. A message is said to be *inconsistent* if it contains some token and its reverse, and *consistent* otherwise. A message w is said to be *vacuous* if, for each token t that it contains, w contains equal numbers of copies of t and $\tilde{t}$. A token t is said to be *effective* for S if $St \neq S$, and a message w is *stepwise effective* for S if each successive token in the sequence of transformations of S by w is effective. A medium is then defined to be a system of states and tokens satisfying the following independent axioms:

(1) Each token has a unique reverse.
(2) For any two distinct states S, Q, there exists a consistent message w with $Sw = Q$.
(3) If message w is stepwise effective for S, then $Sw = S$ if and only if w is vacuous.

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(4) If $Sw = Qz$, w is stepwise effective for S, z is stepwise effective for Q, and both w and z are consistent, then wz is consistent.

The following familiar combinatorial systems can be interpreted as media, although the justification for this claim is not always trivial:

Permutations: The set of permutations of a finite set of items forms the states of a medium, with a token t_{xy} for each ordered pair xy of items that replaces an adjacent pair yx in a permutation by the pair xy, or leaves the permutation unchanged if no such pair exists. The reverse of t_{xy} is t_{yx}.

Topological orderings: For any directed acyclic graph (DAG) G, one can define a medium with states that are the topological orderings of G, and the same swap operations t_{xy} as above for each pair of vertices not connected by a directed path. When G has no edges, we get the permutation medium on the vertices of G.

Acyclic orientations: Let G be an undirected graph, define a state to be an acyclic orientation of the edges of G, and for any ordered pair xy of adjacent vertices define a token t_{xy} that reorients edge (x, y) from x to y, if the resulting orientation is acyclic, and leaves the orientation unchanged otherwise. The result is a medium. When G is complete, the medium of acyclic orientations of G reduces to the permutation medium on the vertices of G. When G is complete bipartite, the medium of acyclic orientations of G reduces to the medium of biorders from the vertices on one side of the bipartition to the vertices on the other side [9].

Hyperplane arrangements: Let C be an open convex region in $\mathbb{R}^d$, and A be a hyperplane arrangement in C. Then the convex chambers of A form the states of a medium [28], with one token t_h for each open halfspace h bounded by a hyperplane in A. If chamber S is included in $C \backslash h$ and shares a facet with a chamber S' included in h, then $St_h = S'$; otherwise $St_h = S$. In the special case where $C = \mathbb{R}^n$, with Cartesian coordinates x_i, and A is the arrangement of hyperplanes $x_i = x_j$ for the edges (i, j) of an n-vertex graph, this is isomorphic to the acyclic orientation medium described above [26, Lemma 2.93]. We can also realize the medium of topological orderings by using hyperplanes $x_i = x_j$ for all i and j, with C consisting of the points in $\mathbb{R}^d$ satisfying $x_i \leqslant x_j$ for each arc (i, j) in the given DAG.

Face-regular graphs: Chepoi et al. [7] discuss center and diameter algorithms for several interesting families of planar graphs, which they call *square systems*, *hexagonal systems*, and *squaregraphs*. The first two families are graphs formed by the portion of a regular tiling of the plane contained within a simple closed curve; a squaregraph is a planar graph in which all faces (except for the outer ones) are quadrilaterals, and in which each interior vertex has degree at least 4. The algorithms of Chepoi et al. rely on simple data structures for reporting distances in these graphs, based on isometric embeddings of the graphs into products of two or three trees. More generally, any graph that can be isometrically embedded into a product of any number of trees is an *isometric hypercube subgraph* or *partial cube*: that is, its vertices can be embedded into a hypercube in such a way that graph distance equals L_1 hypercube distance. We can form a medium from any such embedding by defining the states to be the vertices of the graph, and by letting the action of each token change one coordinate of a vertex within the hypercube, if the resulting vertex belongs to the graph, and by leaving the vertex unchanged otherwise. Conversely, any medium can be induced by an isometric hypercube subgraph in this way. Isometric hypercube subgraphs can be recognized (and an embedding constructed) in time $O(nm)$ (where n denotes the number of vertices of the graph, and m denotes the number of edges) using Djokovic's characterization of these graphs [2,8]; our focus in this paper is rather on what algorithms can be performed efficiently once such a representation is known.

Well-graded families of sets: A family $\mathscr{W}$ of subsets of a finite set $X = \cup\mathscr{W}$ is *well-graded* [9] if, for any two distinct sets S and Q in $\mathscr{W}$, there exists a sequence $S = S_0, S_1, \ldots, S_k = Q$ in $\mathscr{W}$ such that $|S_{j-1}\Delta S_j| = 1$ for $1 \leqslant j \leqslant k = |S\Delta Q|$. (Here, Δ denotes the set-theoretic symmetric difference.) Falmagne and Ovchinnikov [17] show that such a family can be cast as a medium, the states of which are the sets in $\mathscr{W}$. To each x in X are associated two tokens i_x and d_x, defined, respectively, by: $Si_x = S \cup \{x\}$ if $S \cup \{x\} \in \mathscr{W}$, and $Si_x = S$ otherwise; and $Sd_x = S\backslash\{x\}$ if $S\backslash\{x\} \in \mathscr{W}$, and $Sd_x = S$ otherwise. It is easily verified that the set of states $\mathscr{W}$ and the collection of tokens i_x and d_x satisfy the axioms of a medium. Notice that for each x in X, the tokens i_x and d_x are mutual reverses. As suggested by the remaining examples in this section, this structure subsumes many special cases.

Downward closed set families: Let $\mathscr{F}$ denote a family of subsets of a finite set $X = \cup\mathscr{F}$ having the property that $\mathscr{F}$ contains any subset of any of its members. For instance, the independent sets in a graph or matroid define such a family. Then $\mathscr{F}$ is well-graded and so can be represented as a medium.

Binary trees: One can encode a binary tree as a set of integers, that give the heap numbers of nodes in the tree: the root has number 0, and the left and right children of a node with number i have numbers $2i + 1$ and $2i + 2$. Let $\mathscr{F}$