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Games played by Boole and Galois☆

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Abstract

We define an infinite class of 2-pile subtraction games, where the amount that can be subtracted from both piles simultaneously is an extended Boolean function f of the size of the piles, or a function over GF(2). Wythoff's game is a special case. For each game, the second player winning positions are a pair of complementary sequences. Sample games are presented, strategy complexity questions are discussed, and possible further studies are indicated. The motivation stems from the major contributions of Professor Peter Hammer to the theory and applications of Boolean functions.

Keywords: 2-Pile subtraction games; Extended Boolean functions; Galois field; Integer sequences

1. Introduction

We invented 2-pile Boolean subtraction games to pay tribute to Peter Hammer, in honor of his outstanding scientific achievements, in particular his major contributions to the theory and applications of Boolean and pseudo-Boolean functions. The applications Peter has contributed to span a very wide spectrum of human activity, including optimization, maximization, minimization, operations research; and lately, medical applications, about which Peter lectured in his captivating invited address at the workshop.

Within the class of 2-player perfect information games without chance moves, we consider games on two piles of tokens (x, y) of sizes x, y, with $0 \leqslant x \leqslant y < \infty$. Their interest stems, inter alia, from the special and important case of Wythoff's game [20]. See also [1–7,11,12,16–18,21].

For any acyclic combinatorial game, such as 2-pile subtraction games, a position $u = (x, y)$ is labeled N (*Next* player win) if the player moving from u can win; otherwise it is a P-position (*Previous* player win). Denote by $\mathscr{P}$ the set of all P-positions, by $\mathscr{N}$ the set of all N-positions, and by $F(u)$ the set of all (direct) *followers* or *options* of u. It is easy to see that for any acyclic game,

$$u \in \mathscr{P} \quad \text{if and only if} \quad F(u) \subseteq \mathscr{N}, \tag{1}$$

$$u \in \mathscr{N} \quad \text{if and only if} \quad F(u) \cap \mathscr{P} \neq \emptyset. \tag{2}$$

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Indeed, player I, beginning from an N-position, will move to a P-position, which exists by (2), and player II has no choice but to go to an N-position, by (1). Since throughout our games are finite and acyclic, player I will eventually win by moving to a leaf, which is clearly a P-position.

The partitioning of the game's positions into the sets $\mathscr{P}$ and $\mathscr{N}$ is unique for every finite acyclic combinatorial game without ties.

In our games, two players alternate removing tokens from the piles:

(a) Remove any positive number of tokens from a single pile, possibly the entire pile.
(b) Remove a positive number of tokens from each pile, say k, ℓ, so that $|k-\ell|$ is not too large with respect to the position (x_1, y_1) moved to from (x_0, y_0), namely, $|k-\ell| < f(x_1, y_1, x_0)$, equivalently:

$$|(y_0-y_1)-(x_0-x_1)| = |(y_0-x_0)-(y_1-x_1)| < f(x_1, y_1, x_0), \tag{3}$$

where the *constraint function* $f(x_1, y_1, x_0)$ is integer-valued and satisfies:

- Positivity:

$$f(x_1, y_1, x_0) > 0 \quad \forall y_1 \geqslant x_1 \geqslant 0 \ \forall x_0 > x_1.$$

- Monotonicity:

$$x_0' < x_0 \Longrightarrow f(x_1, y_1, x_0') \leqslant f(x_1, y_1, x_0).$$

- Semi-additivity (or generalized triangle inequality) on the P-positions (A_i, B_i) $(A_i \leqslant B_i$ for all $i \geqslant 0)$, namely: for $n > m \geqslant 0$,

$$\sum_{i=0}^{m} f(A_{n-1-i}, B_{n-1-i}, A_{n-i}) \geqslant f(A_{n-m-1}, B_{n-m-1}, A_n).$$

The player making the move after which both piles are empty (a *leaf* of the game) wins; the opponent loses.

Let $S \subset \mathbb{Z}_{\geqslant 0}$, $S \neq \mathbb{Z}_{\geqslant 0}$, and $\overline{S} = \mathbb{Z}_{\geqslant 0} \setminus S$. The *minimum excluded value* of S is

$$\operatorname{mex} S = \min \overline{S} = \text{least nonnegative integer not in} S.$$

Note that mex of the empty set is 0.

We defined the above class of games in [10], where we proved:

Theorem 1. *Let* $\mathscr{S} = \bigcup_{i=0}^{\infty}(A_i, B_i)$, *where, for all* $n \in \mathbb{Z}_{\geqslant 0}$,

$$A_n = \operatorname{mex}\{A_i, B_i : 0 \leqslant i < n\}, \tag{4}$$

$B_0 = 0$, *and for all* $n \in \mathbb{Z}_{>0}$,

$$B_n = f(A_{n-1}, B_{n-1}, A_n) + B_{n-1} + A_n - A_{n-1}. \tag{5}$$

If f is positive, monotone and semi-additive, then $\mathscr{S}$ *is the set of P-positions of a general* 2*-pile subtraction game with constraint function f, and the sequences* $A = \bigcup_{i=1}^{\infty}\{a_i\}$, $B = \bigcup_{i=1}^{\infty}\{b_i\}$ *share the following common features*: (i) *they partition* $\mathbb{Z}_{\geqslant 1}$; (ii) $b_{n+1} - b_n \geqslant 2$ *for all* $n \in \mathbb{Z}_{\geqslant 0}$; (iii) $a_{n+1} - a_n \in \{1, 2\}$ *for all* $n \in \mathbb{Z}_{\geqslant 0}$.

We also showed there that if any of the three conditions of Theorem 3 is dropped, then there are games for which its conclusion fails:

Proposition 1. *There exist* 2*-pile subtraction games with constraint functions f which lack precisely one of positivity, monotonicity or semi-additivity, such that* $\mathscr{S} \neq \mathscr{P}$, *where* $\mathscr{S} = \bigcup_{i=0}^{\infty}(A_i, B_i)$, *and* A_i *satisfies* (4) $(i \in \mathbb{Z}_{\geqslant 0})$; $B_0 = 0$, B_n *satisfies* (5) $(n \in \mathbb{Z}_{>0})$.