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Hybrid one-dimensional reversible cellular automata are regular

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Abstract

It is shown that the set of hybrid one-dimensional reversible cellular automata (CA) with the periodic boundary condition is a regular set. This has several important consequences. For example, it allows checking whether a given CA is reversible and the random generation of a reversible CA from the uniform distribution, both using time polynomial in the size of the CA. Unfortunately, the constant term in the resulting random generation algorithm is much too large to be of practical use. We show that for the less general case of null boundary (NB) CA, this constant can be reduced drastically, hence facilitating a practical algorithm for uniform random generation. Our techniques are further applied asymptotically to count the number of reversible NBCA.

Keywords: Reversible cellular automaton; Hybrid cellular automaton; Finite state automaton; Regular language

1. Introduction

Cellular automata (CA) are a class of discrete dynamical systems that have been applied to model a wide range of scientific phenomena, generate random data, perform computation, and many other applications [12,18,5,17]. A CA can be described as a set of *cells* embedded on a lattice, each of which can exist in a finite set of states. The system evolves by all cells updating their state according to some function of the states of the cells in a local neighborhood; this function is called the cell's *rule*. In this manner, a global configuration (i.e. mapping of cells to states) is taken to a successor configuration by the simultaneous update of all cells. This paper is concerned with *reversible* CA, which are CA having invertible global successor functions.

The class of CA we consider are called *hybrid*, meaning that each cell may employ a different rule for determining its next state. This contrasts with the *uniform* CA popularized by, e.g., Conway [4] and Wolfram [18], in which all cells use the same rule. Our CA are also characterized as one-dimensional (meaning the cells are embedded on a one-dimensional lattice), finite (referring to the number of cells), and nearest neighbor (meaning that each cell interacts with only its left and right neighbors). As our CA are finite and one-dimensional, they are naturally expressed as a finite string of rules.

In this paper, we show that the *reversible* CA form a regular set of strings. This result generalizes a previous result of Sarkar and Barua [15], which showed that hybrid CA involving only the two linear rules 90 and 150 are regular. We

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show that allowing *any* of the 256 possible nearest neighbor rules still preserves regularity. As in Sarkar and Barua's paper, we also show regularity holds regardless of which CA boundary condition is used.

Two important consequences of our regularity result are that there exists a linear time reversibility test algorithm, and that there exists a polynomial time algorithm that generates a random reversible CA from the uniform distribution. Unfortunately, only the former algorithm can be implemented in practice. This is because our regularity result is obtained by defining a nondeterministic automaton for the *complement* of the reversible CA, i.e. the language of irreversible CA. To obtain an automaton that accepts the reversible CA, we must determinize and complement, which causes an exponential blow up. The resulting automaton has $2^{(2^9)}$ states. This is not a show-stopper for the reversibility test algorithm, since it does not explicitly construct the entire automaton. On the other hand, the random generation algorithm must visit all $2^{(2^9)}$ states. In summary, the implied random generation algorithm may theoretically use polynomial time, but the hidden constant is astronomical.

To solve this problem, and hence produce a practical algorithm for uniform random generation, we focus on the null boundary (NB) condition. For this case, we show how the vast nondeterministic automaton for the more general periodic boundary (PB) condition can be reduced to a deterministic automaton having a mere nine states. This allows us to construct a *practical* algorithm that generates a random reversible CA from the uniform distribution. We have in fact implemented this algorithm; it generates a reversible CA with 200 cells in 90 sec on a contemporary laptop computer. The succinctness and determinism of the reduced automaton also allows us to *count* the number of NB reversible CA. We show that the number of such CA with n cells is $\Theta(\lambda^n)$, where $\lambda \approx 17.98$ (whereas there are 256^n possible CA with n cells).

The paper is organized as follows. In Section 2 we further touch on related work. Section 3 lays out the definitions and terminology used throughout the paper. Section 4 shows our main result on the regularity of PB reversible CA. Section 5 develops a greatly reduced automaton for NB reversible CA, which allows for a practical random generation algorithm in Section 6 and also for counting these CA in Section 7. The paper is concluded in Section 8.

2. Related work

A special class of CA involving only *linear* rules lend themselves to algebraic analysis. Linear rules are those that can be expressed as a modulo-2 sum (i.e. XOR logic). The global transition function of linear CA can be expressed as a matrix over the Galois field of two elements and is invertible if and only if this matrix is non-singular. *Additive* rules generalize linear rules somewhat in that they allow negation; these are explored thoroughly in the book of Pal Chaudhuri et al. [12].

Our results generalize previous results of Sarkar and Barua [15], who show that the set of reversible CA over the two linear rules 90 and 150 is regular, for both NB and PB conditions. Sarkar and Barua observe that the determinant of the transition matrix can be expressed by a multi-variate polynomial known as a *continuant*, which admits a recursive definition. Using this recursive definition, they obtain an inductive definition of all (90, 150) CA that yield a non-zero determinant, and hence are reversible. We note that this approach depends on the linearity of the rules 90 and 150; since we allow for nonlinear rules, we require a different technique. Sarkar and Barua also show that roughly 2/3 (resp. 1/3) of all null (resp. periodic) boundary CA with these two rules are reversible. Interestingly, it follows from our result of Section 7 that the ratio of reversible CA with *any* of the 256 nearest neighbor rules vanishes as n increases.

Our CA can be thought of as imposing the restriction on general n-bit finite state machines that the next state of each bit only depends on itself and its two neighbors. Another natural restriction yields the *feedback shift register* (FSR). Results pertaining to reversibility of FSR have been covered by Golomb [6]. In particular, it is shown that of the $2^{(2^n)}$ possible FSR, $2^{(2^{n-1})}$ are reversible. Golomb also provides several necessary and sufficient conditions for FSR reversibility.

There has been a wealth of work on reversibility of uniform, infinite CA (UICA). An early result by Moore [10] and Myhill [11] is the *Garden-of-Eden Theorem*, which states that UICA is surjective if and only if it is injective when restricted to finite configurations. Another important result is due to Richardson [14], and states that if the global successor function of a UICA is injective, then it must also be bijective (i.e. reversible). Amoroso and Patt have given an algorithm that decides reversibility of the (unique) one-dimensional UICA that uses a given rule [1]. Their algorithm, though not described in the language of automata theory, is similar to ours in that it essentially runs a finite state automaton (FSA) that attempts to construct distinct configurations that have the same successor. Our