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Easy and hard instances of arc ranking in directed graphs

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Abstract

In this paper we deal with the arc ranking problem of directed graphs. We give some classes of graphs for which the arc ranking problem is polynomially solvable. We prove that deciding whether $\chi'_r(G) \leqslant 6$, where G is an acyclic orientation of a 3-partite graph is an NP-complete problem. In this way we answer an open question stated by Kratochvil and Tuza in 1999.

Keywords: Graph ranking; Digraph; Computational complexity; Caterpillar

1. Introduction

An *edge k-ranking* of a simple graph is a coloring of its edges with k colors such that each path connecting two edges with the same color contains an edge with a bigger color. Parallel assembly of multipart products from their components is an example of a potential application of the edge ranking problem [2,3]. In the case of the edge ranking of trees the first result was given in [4] where an $O(n \log n)$ time approximation algorithm with a worst case performance ratio of 2 was described. Now, a linear time algorithm is known for optimal edge ranking of trees [8]. On the other hand, this problem remains NP-hard in the case of general graphs [7] of multitrees [2].

A function c mapping the set of vertices of a digraph $G = (V(G), E(G))$ into the set of integers $\{1, \ldots, k\}$ is a *vertex k-ranking* of G if each directed path between two vertices with the same color contains a vertex with a greater color, where a *directed path* connecting vertices u and v is a set of arcs $(uv_1), (v_1v_2), \ldots, (v_{i-1}v_i), (v_iv)$. The symbol $\chi_r(G)$ denotes the smallest number k such that there exists a vertex k-ranking of G. The vertex ranking problem of directed graphs was introduced in [6], where it was shown that it can be solved in polynomial time in the case of oriented trees. On the other hand, deciding whether $\chi_r(G) \leqslant 3$, where G is an acyclic orientation of a planar bipartite graph is an NP-complete problem [6].

In this paper we consider the arc ranking problem of directed graphs. A *directed path* between arcs (uv) and $(u'v')$ is any set of arcs $(v_1v_2), \ldots, (v_{i-1}v_i)$ such that $v_1 \in \{u, v\}$ and $v_i \in \{u', v'\}$, or $v_1 \in \{u', v'\}$ and $v_i \in \{u, v\}$. Then, function $c : E(G) \to \{1, \ldots, k\}$ is an *arc k-ranking* of a digraph G if each directed path connecting arcs with the same color i contains an arc with a color $j > i$. The smallest integer k such that G has an arc k-ranking is denoted by $\chi'_r(G)$.

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Section 2 gives an example of a family of graphs for which the arc ranking problem can be solved efficiently. In particular, a linear time algorithm for optimal coloring of caterpillars is described. This implies that some well-known classes of directed graphs like oriented paths or comets can be colored efficiently. An interesting question is whether the arc ranking problem can be solved in polynomial time for directed trees and we leave it as an open problem. In Section 3 we consider the complexity of the arc ranking problem. For an undirected graph deciding whether there exists an optimal edge ranking using a fixed number of colors can be done in constant time [1]. However, we prove in this paper that the decision problem

- input: G—an acyclic orientation of a 3-partite simple graph,
- question: $\chi'_r(G) \leqslant 6$?

is NP-complete. In this way we answer an open question stated in [6]. Moreover, this result gives a motivation for designing efficient algorithms for some special classes of acyclic digraphs—a nontrivial example is given in the next section.

2. A polynomial time algorithm for caterpillars

A color i is *visible* for $e \in E(G)$ (resp. $v \in V(G)$) if there exists a directed path between e (resp. v) and some arc with color i such that all arcs of this path have smaller colors than i. We say that arc e (vertex v) is *incident* to color i if e (resp. v) is adjacent (resp. incident) to some arc with color i. A *caterpillar* T is a tree containing subgraph P which is a path such that each vertex of T belongs to P or is adjacent to some vertex of P. The vertices of T which belong to P are denoted by $v_0, v_1, \ldots, v_{|V(P)|-1}$ and arcs by $e_1, \ldots, e_{|E(P)|}$, where $e_i = (v_i v_{i-1})$ or $e_i = (v_{i-1} v_i)$ and the arcs e_i, e_{i+1} are adjacent, $i = 1, \ldots, |E(P)| - 1$. The set of arcs in $E(T) \backslash E(P)$ incident to vertex $v_i \in V(P)$ is denoted by $E_i = \{e_i^1, \ldots, e_i^{k_i}\}$. The symbol $\deg_G(v)$ denotes the number of arcs (incoming and outgoing) adjacent to node v in digraph G.

We split P into the set of subpaths $P^1, \ldots, P^l$ such that each P^i is a directed subpath and P^i is not a proper subgraph of any other directed subpath in P. We say that arc e_i is the *first* arc of P^j if $e_i \in E(P^j)$ and $e_{i-1} \notin E(P^j)$. Similarly, e_i is the *last* arc of P^j if $e_i \in E(P^j)$ and $e_{i+1} \notin E(P^j)$. A path P^i is said to be *short* if it contains at most two arcs. Otherwise the subpath is *long*. Fig. 1 depicts an example of a caterpillar.

If G is a digraph and $S \subseteq V(G)$ then the subgraph of G *induced* by S is defined as $G[S] = (S, \{(uv) \in E(G) : u, v \in S\})$. Let $N(v)$ denote the set of neighbors of node v in T. We define

$$T_i = T[\{v_0\} \cup N(v_0) \cup \cdots \cup N(v_{i-1}) \cup (N(v_i) \backslash \{v_{i+1}\})], \quad 0 \leqslant i \leqslant |V(P)|,$$

$$T_{i,j} = T[V(T_j) \backslash V(T_{i-1})], \quad 0 \leqslant i \leqslant j \leqslant |V(P)|,$$

where $V(T_{-1}) = \emptyset$. Assume that we have an arc ranking c of T_i, where e_i is the last arc of some subpath. Define two sets $A_i(c)$, $B_i(c)$ so that $A_i(c)$ contains all colors of arcs which are incident to v_i and $B_i(c)$ contains all colors visible for e_{i+1} which do not belong to $A_i(c)$. In other words, the set $B_i(c)$ contains colors which are forbidden for the arc e_{i+1} and each color s in $A_i(c)$ is forbidden for each arc e of $T_{i,|V(P)|}$ such that e is connected to e_i by a directed path in T and all arcs of this directed path get smaller colors than s. We say that an arc ranking c' of T_j *extends* an arc ranking c of T_i, $j > i$ if c' is valid and $c'|_{E(T_i)} = c|_{E(T_i)}$, i.e. $c'(e) = c(e)$ for each $e \in E(T_i)$. Observe that an arc ranking of

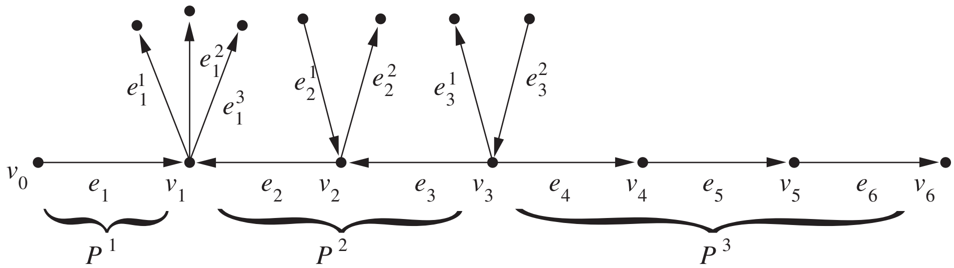

Fig. 1. A caterpillar containing two short subpaths P^1, P^2 and one long subpath P^3.