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Communication

Coloring some classes of mixed graphs

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Abstract

We consider the coloring problem for mixed graphs, that is, for graphs containing edges and arcs. A mixed coloring c is a coloring such that for every edge $[x_i, x_j]$, $c(x_i) \neq c(x_j)$ and for every arc (x_p, x_q), $c(x_p) < c(x_q)$. We will analyse the complexity status of this problem for some special classes of graphs.

Keywords: Mixed graphs; Bipartite graphs; Partial k-trees; Precoloring extension; *SAT*

1. Introduction

Scheduling problems containing incompatibility constraints are very often modelled by undirected graphs: every vertex corresponds to a job and two vertices are joined by an edge if the corresponding jobs cannot be processed at the same period. A vertex coloring of the graph then gives a possible schedule respecting the constraints. In general scheduling problems, there are often more requirements than just incompatibility constraints. Hence the ordinary coloring model is too limited to be useful in many scheduling applications. We will consider here scheduling problems containing incompatibility and precedence constraints: several pairs of jobs have to be processed in a given order. To handle these problems, we have to introduce a more general model, able to take into account these requirements: mixed graphs. These graphs have been introduced for the first time in [11].

A *mixed graph* $G_{\mathrm{M}} = (X, U, E)$ is a graph containing edges (set E) and arcs (set U). An edge joining vertices x_i and x_j will be denoted by $[x_i, x_j]$ and an arc with tail x_p and head x_q by (x_p, x_q). Thus a precedence constraint, saying that job p must be processed before job q, will be represented by an arc (x_p, x_q). A k-coloring of a mixed graph $G_{\mathrm{M}} = (X, U, E)$ is a function c: $X \to \{0, 1, \ldots, k-1\}$ such that for $[x_i, x_j] \in E$, $c(x_i) \neq c(x_j)$ and for $(x_p, x_q) \in U$, $c(x_p) < c(x_q)$. Notice that the mixed graph G_{M} must be acyclic, i.e. must not contain any directed circuit, otherwise no proper k-coloring would exist. Also notice that there is a one-to-one correspondence between a feasible schedule in k time units and a k-coloring of the mixed graph G_{M}. The smallest k such that there exists a k-coloring of G_{M} is called the *mixed chromatic number* and will be denoted by $\gamma(G_{\mathrm{M}})$. Let $G^0_{\mathrm{M}} = (V, U, \emptyset)$ be the directed partial graph of G_{M}. If $\gamma(G^0_{\mathrm{M}})$ denotes the chromatic number of G^0_{M}, that is, the length of a longest directed path in G^0_{M} plus one, then we

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conclude that $\gamma(G_{\mathrm{M}}) \geqslant \gamma(G_{\mathrm{M}}^0)$. In this paper we only consider finite mixed graphs G_{M} containing no directed circuits, no multiple edges or multiple arcs and no loops.

Obviously, coloring the vertices of a mixed graph is more general than the ordinary vertex coloring problem and thus it is *NP*-complete. There is not much literature about mixed graph coloring. In [3], an $\mathrm{O}(n^2)$-algorithm to color optimally mixed trees and bounds on the mixed chromatic number for general mixed graphs are given. For mixed bipartite graphs, the mixed chromatic number is bounded above by $\gamma(G_{\mathrm{M}}^0)+1$ and hence can only take two values. In [3] an open question is the complexity to decide whether it is $\gamma(G_{\mathrm{M}}^0)$ or $\gamma(G_{\mathrm{M}}^0)+1$ for mixed bipartite graphs. Rote has shown with an elementary construction that this problem is *NP*-complete [8]. Here, we will strengthen this result by proving that it is *NP*-complete even for planar bipartite graphs and for bipartite graphs with maximum degree 3. In [9,10] the unit-time job-shop problem is considered via mixed graph coloring. In this case, G_{M}^0 is the union of disjoint paths and $(V, \emptyset, E)$ is the union of disjoint cliques. In [10] three branch-and-bound algorithms are developed and tested on randomly generated mixed graphs of order at most 200 for the exact solution and of order at most 900 for the approximate solution. In [12] mixed graph colorings ϕ for which an arc (x_p, x_q) implies that $\phi(x_p) \leqslant \phi(x_q)$ are considered.

In this paper, we will consider some special classes of graphs and analyse the complexity status of the mixed graph coloring problem for these classes.

2. Some complexity results

First, we will give some definitions taken from [3] which we will use throughout this paper.

Definitions: Let $G_{\mathrm{M}}=(X, U, E)$ be a mixed graph. The *inrank* of a vertex x_i, denoted by $\mathrm{in}(x_i)$, is the length of a longest directed path ending at x_i and the *outrank* of x_i, denoted by $\mathrm{out}(x_i)$, is the length of a longest directed path starting at x_i.

We denote by n the number of vertices in a mixed graph $G_{\mathrm{M}}=(X, U, E)$, i.e. $n=|X|$, and by $N(P)$ the number of vertices on a directed path P.

All graph theoretical terms not defined here can be found in [1].

We will give now some complexity results for some special classes of graphs.

Theorem 1. *Let G_{M} be a mixed graph having the following properties*:

(1) *for all $x_i \in X$, there exists $x_j \in X$ such that $(x_i, x_j) \in U$ or $(x_j, x_i) \in U$*;
(2) *for all maximal directed paths P in G_{M}, $N(P)=\gamma(G_{\mathrm{M}}^0)$ or $N(P)=\gamma(G_{\mathrm{M}}^0)-1$.*

Then deciding whether $\gamma(G_{\mathrm{M}})=\gamma(G_{\mathrm{M}}^0)$ or $\gamma(G_{\mathrm{M}})>\gamma(G_{\mathrm{M}}^0)$ can be done in polynomial time.

Proof. We transform the problem into a $2SAT$ problem which is known to be polynomially solvable [2]. Denote by $\mathscr{P}$ the set of vertices belonging to a path P with $N(P)=\gamma(G_{\mathrm{M}}^0)$.

(1) to each vertex $x \in \mathscr{P}$ with $\mathrm{in}(x)=r$, we associate a variable x_r and a clause (x_r);
(2) to each vertex $x \notin \mathscr{P}$ with $\mathrm{in}(x)=r$, we associate two variables x_r and x_{r+1};
(3) to each path $P=(x^0, x^1, \ldots, x^{\gamma(G_{\mathrm{M}}^0)-2})$ with $N(P)=\gamma(G_{\mathrm{M}}^0)-1$, we associate the clauses $(x_i^i \vee x_{i+1}^i)$, $(\bar{x}_i^i \vee \bar{x}_{i+1}^i)$, for $i=0,1,\ldots,\gamma(G_{\mathrm{M}}^0)-2$, and the clause $(\bar{x}_{j+1}^j \vee \bar{x}_{j+1}^{j+1})$, for $j=0,1,\ldots,\gamma(G_{\mathrm{M}}^0)-3$;
(4) to each edge $[x, y] \in E$ such that $x \in \mathscr{P}$, $y \notin \mathscr{P}$ and $\mathrm{in}(x)=\mathrm{in}(y)=r$ (resp. $\mathrm{in}(x)=\mathrm{in}(y)+1=r+1$), we associate the clause $(\bar{x}_r \vee \bar{y}_r)$ (resp. $(\bar{x}_{r+1} \vee \bar{y}_{r+1})$);
(5) to each edge $[x, y] \in E$ such that $x, y \notin \mathscr{P}$ and $\mathrm{in}(x)=\mathrm{in}(y)=r$ (resp. $\mathrm{in}(x)=\mathrm{in}(y)+1=r+1$), we associate the clauses $(\bar{x}_r \vee \bar{y}_r)$, $(\bar{x}_{r+1} \vee \bar{y}_{r+1})$ (resp. $(\bar{x}_{r+1} \vee \bar{y}_{r+1})$);
(6) to each edge $[x, y] \in E$ such that $x, y \in \mathscr{P}$ and $\mathrm{in}(x)=\mathrm{in}(y)=r$, we associate the clause $(\bar{x}_r \vee \bar{y}_r)$.

Suppose that an instance of $2SAT$ is true. If a variable x_r is set to be 'true', then we will color the corresponding vertex x with color r, i.e. $c(x)=r$. Notice that each vertex $x \in \mathscr{P}$ will be colored with $c(x)=\mathrm{in}(x)$ (see (1)) and each vertex $x \notin \mathscr{P}$ will be colored with $c(x)=\mathrm{in}(x)$ or $c(x)=\mathrm{in}(x)+1$ (see (2) and (3)). Thus, the coloring uses at most $\gamma(G_{\mathrm{M}}^0)$ colors. The clauses in (1) and (3) ensure that for all $(x, y) \in U$ we have $c(x)<c(y)$ and the clauses in (1), (4), (5) and (6) ensure that for all $[x, y] \in E$, $c(x) \neq c(y)$. So we conclude that $\gamma(G_{\mathrm{M}})=\gamma(G_{\mathrm{M}}^0)$.