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Note

Every toroidal graph without adjacent triangles is $(4, 1)^*$-choosable

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Abstract

In this paper, a structural theorem about toroidal graphs is given that strengthens a result of Borodin on plane graphs. As a consequence, it is proved that every toroidal graph without adjacent triangles is $(4, 1)^*$-choosable. This result is best possible in the sense that K_7 is a non-$(3, 1)^*$-choosable toroidal graph. A linear time algorithm for producing such a coloring is presented also.

MSC: 05C15; 05C78

Keywords: Triangle; Choosability; Toroidal graph; Linear algorithm

1. Introduction

All graphs considered are finite and simple. A torus is a closed surface (compact, connected 2-manifold without boundary) that is a sphere with a unique handle, and a toroidal graph is a graph embedable in the torus. For a toroidal graph *G*, we still use *G* to denote an embedding of *G* in the torus.

Let $G=(V, E, F)$ be a toroidal graph, where *V*, *E* and *F* denote the sets of vertices, edges and faces of *G*, respectively. We use $N_G(v)$ and $d_G(v)$ to denote the set and number of vertices adjacent to a vertex v, respectively, and use $\delta(G)$ to denote the minimum degree of *G*. A face of an embedded graph is said to be incident with all edges and vertices on its boundary. Two faces are adjacent if they share a common edge. The degree of a face *f* of *G*, denoted also by $d_G(f)$, is the length of the closed walk bounding *f* in *G*. When no confusion may occur, we write $N(v)$, $d(v)$, $d(f)$ instead of $N_G(v)$, $d_G(v)$, $d_G(f)$. A *k*-vertex (or *k*-face) is a vertex (or face) of degree *k*, a k^--vertex (or k^--face) is a vertex (or face) of degree at most *k*, and a k^+-vertex (or k^+-face) is a vertex (or face) of degree at least *k*. For $f \in F(G)$, we write $f=[u_1u_2\ldots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices clockwisely lying on the boundary of *f*. An *n*-face $[u_1u_2u_3\ldots u_n]$ is called an $(m_1, m_2, \ldots, m_n)$-face if $d(u_i)=m_i$ for $i=1, 2, \ldots, n$. An *n*-circuit is a circuit with exactly *n* edges.

In [7], Lebesgue proved a structural theorem about plane graphs that asserts that every 3-connected plane graph contains a vertex of given properties (see of [5, Theorem 2]). There are many analogous results appeared since then [1–3,5,10,14]. In this paper, we consider the structure of toroidal graphs, and prove a Lebesgue type theorem that strengthens a result given by Borodin in [2].

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Theorem 1. *Let G be a connected toroidal graph. Then, one of the following holds*:

(1) *G contains two adjacent* 3-*faces*.
(2) $\delta(G) < 4$.
(3) *G contains two adjacent* 4-*vertices*.
(4) *G contains a* (4, 5, 5)-*face*.

A list assignment of G is a function L that assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An L-coloring with impropriety d for integer $d \geqslant 0$, or simply an $(L, d)^*$-coloring, of G is a mapping ϕ that assigns a color $\phi(v) \in L(v)$ to each vertex $v \in V(G)$ such that v has at most d neighbors colored with $\phi(v)$. For integers $m \geqslant d \geqslant 0$, a graph is called $(m, d)^*$-*choosable*, if G admits an $(L, d)^*$-coloring for every list assignment L with $|L(v)| = m$ for all $v \in V(G)$. An $(m, 0)^*$-choosable graph is simply called m-choosable.

The notion of list improper coloring was introduced independently by Škrekovski [11] and Eaton and Hull [4]. They proved that every planar graph is $(3, 2)^*$-choosable and every outerplanar graph is $(2, 2)^*$-choosable. In [8], it was proved that every plane graph without 4-circuits and l-circuits for some $l \in \{5, 6, 7\}$ is $(3, 1)^*$-choosable.

The distances of two triangles T_1 and T_2 is defined to be the length of a shortest path connecting a vertex of T_1 to a vertex of T_2. Lam et al. [6] showed that every plane graph without triangles of distance less than 2 is $(4m, m)$-choosable. Xu [13,14] proved that every graph, that can be embedded into a surface of non-negative characteristic and contains no triangles of distance zero, is $(4m, m)$-choosable. Wang et al. [12], independently, proved that every plane graph without triangles of distance zero is 4-choosable. Lam et al. [6], and Wang and Lih [12] independently, proposed a conjecture that claims that every plane graph without adjacent triangles is 4-choosable. This conjecture is still open.

In this paper we relax this conjecture and prove, as a consequence of Theorem 1, that every toroidal graph G without adjacent triangles is $(4, 1)^*$-choosable, and we also give a linear time algorithm for producing an $(L, 1)^*$-coloring for an arbitrary given list assignment L with $|L(v)| \geqslant 4$ for every $v \in V(G)$.

Theorem 2. *Let G be a toroidal graph without adjacent triangles. Then G is* $(4, 1)^*$-*choosable*.

Since K_7 is a toroidal graph, and it is not $(L, 1)^*$-choosable for $L(v) = \{1, 2, 3\}$ for each of its vertices v, Theorem 2 is best possible in this sense.

In Section 2, we give the proofs of our theorems. According to the proof of Theorem 2, a linear time algorithm is given in Section 3.

2. Proofs of the theorems

Proof of Theorem 1. Assume to the contrary that the theorem is false. Let G be a connected toroidal graph with the properties that G contains no adjacent 3-faces, $\delta(G) \geqslant 4$, every 4-vertex is adjacent to only 5^+-vertices, and every 3-face is not a (4, 5, 5)-face. The Euler's formula $|V| + |F| - |E| \geqslant 0$ can be rewritten in the following form:

$$\sum_{v \in V(G)} \left\{ \frac{3 \cdot d_G(v)}{10} - 1 \right\} + \sum_{v \in F(G)} \left\{ \frac{d_G(f)}{5} - 1 \right\} \leqslant 0. \tag{1}$$

Let ω be a weight on $V(G) \cup F(G)$ by defining $\omega(v) = (3 \cdot d(v)/10) - 1$ if $v \in V(G)$, and $\omega(f) = (d(f)/5) - 1$ if $f \in F(G)$. Then the total sum of the weights is no more than zero. To prove Theorem 1, we will introduce some rules to transfer weights between the elements of $V(G) \cup F(G)$ so that the total sum of the weights is kept constant while the transferring is in progress. However, once the transferring is finished, we can show that the resulting weight ω' satisfying $\sum_{x \in V(G) \cup F(G)} \omega'(v) > 0$. This contradiction to (1) will complete the proof.

Our transferring rules are as follows:

(R_1) A 4-vertex transfers $\frac{1}{20}$ to each incident 3-face or 4-face.
(R_2) A 5^+-vertex transfers $\frac{7}{40}$ to each incident 3-face.
(R_3) A 5-vertex transfers $\frac{1}{20}$ to each incident 4-face.
(R_4) A 6^+-vertex transfers $\frac{11}{120}$ to each incident 4-face.