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Sparse connectivity certificates via MA orderings in graphs

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Abstract

For an undirected multigraph $G=(V, E)$, let α be a positive integer weight function on V. For a positive integer k, G is called (k, α)-connected if any two vertices $u, v \in V$ remain connected after removal of any pair (Z, E') of a vertex subset $Z \subseteq V - \{u, v\}$ and an edge subset $E' \subseteq E$ such that $\sum_{v \in Z} \alpha(v) + |E'| < k$. The (k, α)-connectivity is an extension of several common generalizations of edge-connectivity and vertex-connectivity. Given a (k, α)-connected graph G, we show that a (k, α)-connected spanning subgraph of G with $O(k|V|)$ edges can be found in linear time by using MA orderings. We also show that properties on removal cycles and preservation of minimum cuts can be extended in the (k, α)-connectivity.

Keywords: Edge-connectivity; Vertex-connectivity; Connectivity certificates; MA orderings; Mixed cuts; Removable cycles; Spanning subgraphs

1. Introduction

Let $G = (V, E)$ stand for an undirected multigraph defined by a pair of a vertex set V and an edge set E, where an edge e with endvertices u and v is denoted by $\{u, v\}$. Let $n = |V|$ and $m = |E|$. The vertex set and edge set of a graph G may be denoted by $V(G)$ and $E(G)$, respectively. A singleton set $\{x\}$ may be simply written as x. For two subsets $X, Y \subset V$ (not necessarily disjoint), $E(X, Y; G)$ denotes the set of edges joining a vertex in X and a vertex in Y, and $d(X, Y; G)$ denotes $|E(X, Y; G)|$. In particular, they may be written as $E(X; G)$ and $d(X; G)$, respectively, if $Y = V - X$. For a subset $F \subseteq E$ (resp., $X \subseteq V$), we denote by $G - F$ (resp., $G - X$) the graph obtained from G by removing the edges in F (resp., the vertices in X together with the edges incident to a vertex in X).

A *mixed cut* in G is defined to be an ordered partition (A, B, Z) of V such that $A \neq \emptyset$ and $B \neq \emptyset$, where Z is allowed to be empty. We say that a mixed cut (A, B, Z) *separates* vertices u and v if one of u and v belongs to A and the other belongs to B. That is, u and v are disconnected in $G - Z - E(A, B; G)$.

Let $\alpha : V \to \mathbf{Z}_+$ be a vertex weight function, where $\mathbf{Z}_+$ denotes the set of positive integers. For a subset $X \subseteq V$, we denote $\alpha(X) = \sum_{v \in X} \alpha(v)$. Given a function $\alpha : V \to \mathbf{Z}_+$, the *size* of a mixed cut (A, B, Z) is defined to be $\alpha(Z) + d(A, B; G)$.

We define local α-connectivity $\lambda_\alpha(u, v; G)$ between two vertices $u, v \in V$ to be the minimum size of a mixed cut (A, B, Z) separating u and v, i.e.,

$$\lambda_\alpha(u, v; G) = \min\{\alpha(Z) + d(A, B; G) |\ \text{mixed cuts } (A, B, Z) \text{ separating } u \text{ and } v\}.$$

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We say that a family of paths connecting two vertices u and v is α-*independent* if they are edge-disjoint and each vertex $u' \in V - \{u, v\}$ is contained in at most $\alpha(u')$ paths of them. By Menger's theorem, it is a simple matter to see that $\lambda_\alpha(u, v; G)$ is equal to the maximum number of α-independent paths connecting u and v. Two vertices u and v are called (k, α)-*connected* if $\lambda_\alpha(u, v; G) \geqslant k$. A graph is called (k, α)-*connected* if any two distinct vertices are (k, α)-connected.

Observe that $\lambda_\alpha(u, v; G)$ is the local edge-connectivity when $\alpha(x) > d(x; G)$, $x \in V$, while it implies the local vertex-connectivity when $\alpha(x) = 1$, $x \in V$. Moreover, our α-connectivity includes some of previous common generalizations [1,6,8] of the edge and vertex connectivities. For a specified subset $T \subseteq V$ of vertices, we say that a family of paths connecting two vertices u and v is T-*independent* if they are edge-disjoint and every element of T is contained in at most one path as an inner vertex. Frank et al. [6] have defined local T-connectivity $\lambda_T(u, v; G)$ as the maximum number of T-independent paths connecting u and v. Observe that $\lambda_T(u, v; G) = \lambda_\alpha(u, v; G)$ when $\alpha(x) = 1$, $x \in T$ and $\alpha(x) > d(x; G)$, $x \in V - T$.

On the other hand, Berg and Jordán [1] have defined local ℓ-mixed connectivity between two vertices u and v by

$$\mu_\ell(u, v; G) = \min\{\ell|Z| + d(A, B; G) \mid \text{mixed cuts } (A, B, Z) \text{ separating } u \text{ and } v\},$$

where $\ell \geqslant 1$ is a specified integer. They call a graph G ℓ-*mixed* p-*connected* if $|V| \geqslant p/\ell + 1$ and $\mu_\ell(u, v; G) \geqslant p$ for all pairs $u, v \in V$. This is an extension of (k, ℓ)-connectivity previously introduced by Kaneko and Ota [8] in the sense that (k, ℓ)-connectivity is equivalent to ℓ-mixed $k\ell$-connectivity (see also [4] for the (k, ℓ)-connectivity). Obviously $\mu_\ell(u, v; G) = \lambda_\alpha(u, v; G)$ when $\alpha(x) = \ell$, $x \in V$.

For the above connectivity notions, sparse spanning subgraphs preserving k-connectivity have been studied extensively [1,3,5,6,14]. To generalize these results for our α-connectivity, we define certificates of a graph as follows. A spanning subgraph H of a graph G is called a (k, α)-*certificate* of G if

$$\lambda_\alpha(u, v; H) \geqslant \min\{\lambda_\alpha(u, v; G), k\} \quad \text{for every vertex pair } u, v \in V.$$

Clearly a (k, α)-certificate of G is (k, α)-connected if so is G. A (k, α)-certificate is called *sparse* if it has $O(kn)$ edges. Finding such sparse certificates can be used as a preprocessing that reduces the size of graphs input for many connectivity algorithms.

2. MA orderings

For a (multi)graph $G = (V, E)$, a total ordering $\sigma = (v_1, v_2, \ldots, v_n)$ of vertices in V is called a *maximum adjacency ordering* (an MA ordering, for short) in G if

$$d(V_{i-1}, v_i; G) \geqslant d(V_{i-1}, v_j; G) \quad \text{for all } i, j \text{ with } 2 \leqslant i < j \leqslant n,$$

where we denote $V_i = \{v_1, v_2, \ldots, v_i\}$ $(1 \leqslant i \leqslant n)$. Such an ordering can be found by choosing an arbitrary vertex as v_1, and choosing a vertex $u \in V - V_i$ that has the largest number of edges between V_i and u as the $(i+1)$th vertex v_{i+1} after choosing the first i vertices $V_i = \{v_1, \ldots, v_i\}$. This procedure can be implemented to run in $O(n+m)$ time by using an appropriate data structure [14]. We start with the following observation which easily follows from the definition.

Observation 1. *Let $G = (V, E)$ be a forest, and $\sigma = (v_1, v_2, \ldots, v_n)$ be an MA ordering of V in G. Then*:

(i) *Each vertex v_j has at most one incident edge that joins v_j and a vertex v_i with $i < j$.*
(ii) *For each tree T in G, $V(T)$ consists of the consecutive vertices $v_i, v_{i+1}, \ldots, v_\ell$. Then by* (i) *any subsequence $v_i, v_{i+1}, \ldots, v_h$ with $h \leqslant \ell$ induces a connected graph from T.*

We define $\mathscr{F}(G, \sigma) = (F_1, F_2, \ldots, F_m)$ to be the following partition of the edge set E. For each $i = 2, \ldots, n$, consider the set $E(V_{i-1}, v_i; G)$ of edges between V_{i-1} and v_i, and let $e_{i,k} \in E(V_{i-1}, v_i; G)$ be the edge that appears as the kth edge when the edges in $E(V_{i-1}, v_i; G)$ are arranged in the order $e_{i,1} = \{v_{j_1}, v_i\}$, $e_{i,2} = \{v_{j_2}, v_i\}, \ldots, e_{i,p} = \{v_{j_p}, v_i\}$, where $1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_p$ holds. By letting

$$F_k = \{e_{2,k}, e_{3,k}, \ldots, e_{n,k}\}, \quad k = 1, 2, \ldots, m \tag{1}$$

(some of $e_{i,k}$ may be void), we have a partition $\mathscr{F}(G, \sigma) = (F_1, \ldots, F_m)$ of E, where possibly $F_j = F_{j+1} = \cdots = F_m = \emptyset$ for some j. By construction of $\mathscr{F}(G, \sigma)$, we easily have the following observation.