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Representation of Partial Traces

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Abstract

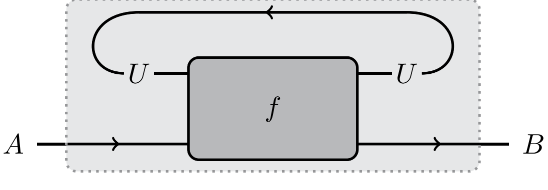
The notion of trace in a monoidal category has been introduced to give a categorical account of a situation occurring in very different settings: linear algebra, topology, knot theory, proof theory... with the trace operation understood as a feedback operation.
Partially traced categories were later introduced to account for cases where the trace is not always defined, and it was shown that partially traced category can always be seen as a subcategory of a totally traced one. We give a new proof of this representation theorem, using a construction that is different from the original one. However, since they satisfy the same universal property they are naturally isomorphic.

Keywords: monoidal category, trace, feedback, representation theorem.

Introduction

Traced monoidal categories were introduced by A. Joyal, R. Street and D. Verity [9] as a common categorical axiomatization of a structure that occurs in very different settings such as linear algebra, topology, knot theory, proof theory... In particular, traced monoidal categories constitute the basis of the categorical approach [1,5] to J.-Y. Girard's *geometry of interaction* program [3].

The basic idea is that a trace is an operation associating to any $f : A \otimes U \to B \otimes U$ in a monoidal category, a new morphism $\mathsf{Tr}^U(f) : A \to B$, this operation being understood as a *feedback along* U, which is acknowledged in the graphical language for these categories [9] by depicting $\mathsf{Tr}^U(f)$ as



This operation has to satisfy a number of axioms that capture formally what is expected of such a notion of feedback.

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More recently, E. Haghverdi and P. J. Scott [6] introduced the notion of *partial trace*, accounting for the fact that the trace operation can be only partially defined. This is a situation that occur very naturally in practice: think of the trace in infinite-dimensional Hilbert spaces [1] or feedback loops in synchronous circuits, for instance. Later on, O. Malherbe, P. J. Scott and P. Selinger [11] showed a representation theorem for partial traces, relating them to total traces: any partially traced category embeds in a totally traced one, with an embedding reflecting the partial trace. Their construction is based on P. Freyd's paracategories [7] and a partial version of the $\mathbf{Int}(\cdot)$ construction [9]. It enjoys a universal property factoring any functor reflecting the partial trace structure.

In this article, we will give a new proof of this result, via a different construction based on tools used in the categorical approach to equivalence of automata [2]. Our proof is more straightforward and does not rely on a delicate argumentation about partially defined operations. As a consequence, the formulation of the universal property we obtain is also more direct as it does not involve compact closed categories as an intermediate step. However, the two constructions satisfy in the end the same universal property that can be rephrased as being left adjoint to the identity functor (they are "free constructions"). Because adjoints are unique up to natural isomorphism, the two construction must be naturally isomorphic.

Outline of the article

In section 1 we fix the notations and vocabulary used in the rest of the article; we recall the definition of a partially traced category, and the associated notion of traced functor.

We then introduce in section 2 the key ingredient of our construction: the *dialect* construction, that allows morphisms to have private interfaces. With this construction comes a *hiding* operation which sets the basis for total extensions of partial traces.

A congruence is then defined in section 3 and its interplay with the monoidal and traced structures is explored. We will show that quotienting the dialect category by this equivalence turns the hiding operation into a total trace (section 3.1).

In section 3.2 and section 4.1, we will show that the original partially traced category embeds in this quotiented category, via an embedding reflecting the partial trace. Finally, we show that our construction enjoys the expected universal property in section 4.2.

1 Partially traced categories

We begin by setting notations and recalling some background definitions.

Notation 1.1 We write the composition of morphisms in a category in the usual order, omitting the $\circ$ symbol: if we have $f : A \to B$ and $g : B \to C$ then

[1] Although this case actually fails to satisfy axiom (v) of definition 1.5 and would need a relaxing of the framework (and a non-trivial, if possible, adaptation of our proof of the representation theorem) to be considered as a partial trace in the categorical sense.