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Towards Compositional Graph Theory

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Dedicated to the memory of R.F.C. Walters.

Abstract

Decomposing graphs into simpler graphs is one of the central concerns of graph theory. Investigations have revealed deep concepts such as *modular decomposition*, *tree width* or *rank width*, which measure—in different ways—the *structural complexity* of a graph's topology. Courcelle and others have shown that such concepts can be used to obtain efficient algorithms for families of graphs that are amenable to decomposition (e.g. those that have bounded tree-width). These algorithms, in turn, are of course of use in computer science, where graphs are ubiquitous. In this paper we take the first steps towards understanding notions of decomposition in graph theory *compositionally*, and more generally, in a categorical setting: category theory, after all, is the mathematics of compositionality.

We introduce the concept of $\cup-matrices$ (cup-matrices). Like ordinary matrices, $\cup$-matrices are the arrows of a PROP: we give a presentation, extending the work of Lafont, and Bonchi, Zanasi and the second author. A variant of $\cup$-matrices is then used in the development of a novel algebra of simple graphs, the lingua franca of graph theory. The algebra is that of a certain symmetric monoidal theory: $\cup$-matrices—akin to adjacency matrices—encode the graphs' topology.

Keywords: Graph decomposition, simple graphs, modular decomposition, (rank) width, cup-matrices

1 Introduction

When category theorists talk about the category of graphs, they usually mean the presheaf category $\mathbf{Graph} = \mathbf{Set}^{\cdot \rightrightarrows \cdot}$. The objects of this category, however, are not what graph theorists would typically refer to as *graphs* tout court – rather, as *directed multigraphs*, because there may be more than one directed edge between any two vertices. The most basic—and important—objects of study in graph theory are *simple graphs*: these are undirected with at most one edge between any two vertices, and no self loops. In this paper, we are concerned with simple graphs.

A *structural metric* of a graph is a way of assigning a numerical value, typically a natural number, to a graph. The intention is for the number to indicate, in some way, the graph's inherent *structural complexity*. Some well-known structural metrics in graph theory include path-width, tree-width, branch-width, clique-width and rank-width. The notion of tree-width is perhaps the best known in Computer Science through the work of Courcelle [6] who showed that monadic second-order

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logic can be decided in linear time for families of graphs with *bounded* tree-width. Courcelle's theorem has found several algorithmic applications (see e.g. [13]).

The structural metric *rank-width*, due to Oum and Seymour, has been a hot topic in graph theory over the last ten years and is of particular relevance for us. We refer to [17, 18] for the technical details, here we give an intuitive description. A *rank-decomposition* of a simple graph with vertex set V can be considered as a binary tree, where the tree-nodes are labelled with nonempty subsets of V and the tree-edges are labelled with natural numbers, such that:

- the labels (vertex sets W_1, W_2) of the children w_1, w_2 of any tree-node w are a (binary) partition of the label (vertex set W) of the parent tree-node: i.e. $W = W_1 \cup W_2$ and $W_1 \cap W_2 = \varnothing$,
- the root is labelled with V,
- the leaves are labelled with singletons,
- the tree-edge from a parent to a child labelled with vertex set W is labelled with the *rank* of the $|V \backslash W| \times |W|$ adjacency $\mathbb{Z}_2$-matrix that tabulates the edges from W to the remainder of the graph. Note that matrix algebra is performed over the field $\mathbb{Z}_2$, i.e. $1 + 1 = 0$.

The *width* of a particular rank-decomposition is its maximum edge label. The *rank-width* of a graph is then the width of an optimal rank-decomposition: one with the smallest width. Discrete graphs and cliques both enjoy a rank-width of 1.

The concepts of rank-width and other structural metrics have proved to be very important in graph theory and related areas. There are two shortcomings, however, where category theory can help:

- *Generality.* The definition, as stated, is specialised to simple graphs. Yet, the underlying concept is quite robust and can be stated mutatis mutandis for other kinds of graphical structures: multigraphs, directed graphs (bi-rank-width), hypergraphs, Petri nets etc. This suggests that the fundamental theory ought to be done in a more general setting.
- *Compositionality.* The notion of a rank-decomposition (and equivalent notions for other structural metrics) is not inherently *compositional* in the sense that knowing how to decompose a graph G may not help in constructing decompositions of a graph H that has G as a sub-component. Intuitively speaking, rank-decompositions forget too much: only rank is recorded – the adjacency information ought to be recorded as well, in some form.

What, then, is a fruitful way of treating simple graphs categorically, in a way that will lead us to understanding structural metrics generally and compositionally? Graphs as objects and homomorphisms as arrows is the traditional approach, yet it does not immedietaly yield an *algebra* of graphs: for that, we need graphs to be the arrows. Cospans are a standard technique for turning objects into arrows, and the algebra of cospans of directed graphs was considered in [9], which is close in spirit to our work, although we do not consider cospans.

Similarly to Fiore and Campos' work on an algebra of directed acyclic graphs [8],