An Analytic Expression for the Binormal Partial Area under the ROC Curve

Stephen L. Hillis, PhD, Charles E. Metz, PhD*

Rationale and Objectives: The partial area under the receiver operating characteristic (ROC) curve (pAUC) is a useful summary measure for diagnostic studies. Unlike most summary measures that are functions of the ROC curve, researchers have not been aware of an analytic expression available for computing the pAUC for an ROC curve based on a latent binormal model. Instead, the pAUC has been computed using numerical integration or a rational polynomial approximation. Our purpose is to provide analytic expressions for the two forms of pAUC.

Materials and Methods: We discuss the two fundamentally different types of pAUC. We present analytic expressions for both types, provide corresponding proofs, and illustrate their application with an example comparing the performances of spin echo and cine magnetic resonance imaging for detecting thoracic aortic dissection.

Results: We provide an example of using the pAUC as the outcome in a multireader multicase analysis. We find that using the pAUC results in a more significant finding.

Conclusions: We have provided analytic expressions for both types of pAUC, making it easier to compute the pAUCs corresponding to binormal ROC curves.

Key Words: Diagnostic radiology; partial area under the ROC curve (pAUC); receiver operating characteristic (ROC) curve.

INTRODUCTION

In diagnostic radiology, receiver operating characteristic (ROC) curves are commonly used to quantify the accuracy with which a reader (typically a radiologist) can discriminate between images from nondiseased (or normal) and diseased (or abnormal) cases. Although the ROC curve concisely describes the tradeoffs between sensitivity and specificity, typically accuracy is summarized by a summary index that is a function of the ROC curve. Commonly used summary indices include the area under the ROC curve (AUC), the partial area under the ROC curve (pAUC), sensitivity for a given specificity, and specificity for a given sensitivity. See Zou et al (1) for a concise introduction to ROC analysis.

A common method for estimating the ROC curve is likelihood estimation under the assumption of a latent binormal model (2–5); alternatively, a generalized linear model approach can also be used (6,7) based on the binormal model assumption. Under the latent binormal model assumption the ROC curve can be described by two parameters. Except for the pAUC, analytic expressions have been routinely employed for expressing the indices previously mentioned as a function of the binormal ROC curve parameters. It is generally believed that the pAUC, assuming a latent binormal model, cannot be expressed as an analytic expression. For example, Pepe (8) states: "Unfortunately, a simple analytic expression does not exist for the pAUC summary measure. It must be calculated using numerical integration or a rational polynomial approximation." Similarly, Zhou et al (9) state: "This partial area as it is known, is evaluated by numerical integration (McClish, 1989)." Although these methods can be programmed, having a simple expression for the pAUC would be much more convenient.

It is generally not known that Pan and Metz (10) provided analytic expressions for the two forms of pAUC. However, the expressions they provided were incorrect and they did not provide proofs for their results. More importantly, it is generally not known that Thompson and Zucchini (11) provided a correct analytic expression for one form of pAUC as well as the proof. In fact, we only became aware of this latter result during the final stage of submitting this article. The purpose of this article is to bring to the attention of the reader the result provided by Thompson and Zucchini, extend their

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*Deceased.

From the Departments of Radiology and Biostatistics, University of Iowa, 3170 Medical Laboratories, 200 Hawkins Drive, Iowa City, IA 52242-1077, and Comprehensive Access and Delivery Research and Evaluation (CADRE) Center, Iowa City VA Health, Iowa City, IA (S.L.H.); Department of Radiology, University of Chicago, Chicago, IL (C.E.M.). Received September 6, 2012; accepted September 19, 2012. Stephen Hillis was supported by the National Institute of Biomedical Imaging and Bioengineering grants R01EB000863 and R01EB013667. **Address correspondence to:** S.L.H. e-mail: steve-hillis@uiowa.edu

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result to the second form of pAUC, and illustrate use of both pAUC expressions with a real data set that compares the relative performance of single spin-echo magnetic resonance imaging (SE MRI) to cinematic presentation of MRI (CINE MRI) for the detection of thoracic aortic dissection. In addition, we provide proofs for both results that are more accessible to radiology researchers and clinicians than the proof given by Thomas and Zucchini.

MATERIALS AND METHODS

Two Different pAUCs

Let FPF and TPF denote false- and true-positive fractions for a given classification threshold such that an image with a test result equal or greater than the threshold is classified as diseased, and otherwise nondiseased. That is, FPF is the probability that a test result for a nondiseased subject exceeds the threshold and TPF is the probability that a test result for a diseased subject exceeds the threshold. The ROC curve is a plot of TPF versus FPF for all possible thresholds. FPF and TPF are the same as 1− specificity and sensitivity, respectively.

Two fundamentally different partial areas have been proposed (11–13). One partial area corresponds to the area under a ROC curve over an interval $(\mathrm{FPF}_1<\mathrm{FPF}_2)$, which we denote by $\mathrm{pAUC}_{\mathrm{FPF}}(\mathrm{FPF}_1, \mathrm{FPF}_2)$. This pAUC is illustrated in Figure 1a and 1b. Often this pAUC is normalized by dividing by $\mathrm{FPF}_2-\mathrm{FPF}_1$, which allows it to be interpreted as the average value of TPF over all values of FPF between FPF_1 and FPF_2. This partial area is typically useful when a clinical task demands high specificity; for this situation, $\mathrm{FPF}_1 = 0$, FPF_2 is small (eg, 0.10, 0.20) and thus it is $\mathrm{pAUC}_{\mathrm{FPF}}(0, \mathrm{FPF}_2)$ that we are interested in computing. Because $\mathrm{pAUC}_{\mathrm{FPF}}(\mathrm{FPF}_1, \mathrm{FPF}_2) = \mathrm{pAUC}_{\mathrm{FPF}}(0, \mathrm{FPF}_2) - \mathrm{pAUC}_{\mathrm{FPF}}(0, \mathrm{FPF}_1)$ for $\mathrm{FPF}_1 < \mathrm{FPF}_2$, it suffices to provide a general formula only for $\mathrm{pAUC}_{\mathrm{FPF}}(0, \mathrm{FPF}_0)$. Walter (14) has discussed using this pAUC with summary ROC curves.

The other pAUC corresponds to the area to the right of the ROC curve in the interval $(\mathrm{TPF}_1<\mathrm{TPF}_2)$, which we denote by $\mathrm{pAUC}_{\mathrm{TPF}}(\mathrm{TPF}_1, \mathrm{TPF}_2)$. This pAUC is illustrated in Figure 1c and 1d. Often this pAUC is normalized by dividing by $\mathrm{TPF}_2 - \mathrm{TPF}_1$, which allows it to be interpreted as the average value of 1 − FPF (ie, specificity) over all values of TPF between TPF_1 and TPF_2. This pAUC is typically useful when a clinical demands high sensitivity: TPF_1 is large, $\mathrm{TPF}_2 = 1$, and thus it is $\mathrm{pAUC}_{\mathrm{TPF}}(\mathrm{TPF}_1, 1)$ that we are interested in computing. Because $\mathrm{pAUC}_{\mathrm{TPF}}(\mathrm{TPF}_1, \mathrm{TPF}_2) = \mathrm{pAUC}_{\mathrm{TPF}}(\mathrm{TPF}_1, 1) - \mathrm{pAUC}_{\mathrm{TPF}}(\mathrm{TPF}_2, 1)$ for $\mathrm{TPF}_1<\mathrm{TPF}_2$, it suffices to provide a general formula only for $\mathrm{pAUC}_{\mathrm{TPF}}(\mathrm{TPF}_0, 1)$.

Analytic Expressions for the pAUCs

In this section, we present analytic expressions for the two forms of pAUC under the assumption of a latent binormal model. These expressions are the primary contribution of this paper. Corresponding proofs are presented in the Appendix.

Binormal model assumptions. Throughout this article, we assume that the ROC curve is based on a latent binormal model. The latent binormal model assumes that the latent decision variable used to classify cases (or some unknown strictly increasing transformation of it) arises from a pair of normal densities corresponding to the nondiseased and diseased case populations, having generally different means and standard deviations. Because ROC curves are invariant under strictly increasing transformations of the decision variable, we can assume without loss of generality that the normal distribution for nondiseased cases has zero mean and unit standard deviation, whereas that for diseased cases has mean μ and standard deviation σ, where $\mu>0$ and $\sigma>0$. Thus letting X and Y denote independent decision variables having the same distributions as the decision variable distributions for nondiseased and diseased cases, respectively, we are assuming that $X \sim N(0, 1)$ and $Y \sim N(\mu, \sigma^2)$.

Results for $pAUC_{FPF}(0, FPF_0)$. Let $\Phi(u)$ denote the standardized normal distribution function; ie, $\Phi(u) = \Pr(U<u)$ where U has a normal distribution with zero mean and unit variance. Let $\mathrm{F}_{\mathrm{BVN}}(z, u; \rho)$ denote the standardized bivariate normal distribution function with correlation ρ; ie, $\mathrm{F}_{\mathrm{BVN}}(z, u; \rho) = \Pr(Z<z \text{ and } U<u)$, where Z and U jointly have a standardized bivariate normal distribution and $\rho = \mathrm{corr}(Z, U)$. This function is available in many statistical software programs, such as SAS, Stata, SPSS, and the freely available R program.

Assuming the binormal model described in the Binormal model assumptions section, $\mathrm{pAUC}_{\mathrm{FPF}}(0, \mathrm{FPF}_0)$ is given by

$$\mathrm{pAUC}_{\mathrm{FPF}}(0, \mathrm{FPF}_0) = \mathrm{F}_{\mathrm{BVN}}\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \Phi^{-1}(\mathrm{FPF}_0); -1/\sqrt{1+\sigma^2}\right) \quad (1)$$

In terms of the binormal parameters $a = \mu/\sigma$ and $b = 1/\sigma$, we can write Equation 1 in the form

$$\mathrm{pAUC}_{\mathrm{FPF}}(0, \mathrm{FPF}_0) = \mathrm{F}_{\mathrm{BVN}}\left(\frac{a}{\sqrt{1+b^2}}, \Phi^{-1}(\mathrm{FPF}_0); -b/\sqrt{1+b^2}\right) \quad (2)$$

Equation 2 is also given by Thompson and Zucchini (11).

Results for $pAUC_{TPF}(TPF_0, 1)$. Assuming the binormal model described in the Binormal model assumptions section, $\mathrm{pAUC}_{\mathrm{TPF}}(\mathrm{TPF}_0, 1)$ is given by

$$\mathrm{pAUC}_{\mathrm{TPF}}(\mathrm{TPF}_0, 1) = \mathrm{F}_{\mathrm{BVN}}\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \Phi^{-1}(1-\mathrm{TPF}_0); -\sigma/\sqrt{1+\sigma^2}\right) \quad (3)$$

In terms of the binormal parameters $a = \mu/\sigma$ and $b = 1/\sigma$, we can write Equation 3 in the form