Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Electronic Notes in Theoretical Computer Science 202 (2008) 191–200
Electronic Notes in Theoretical Computer Science
www.elsevier.com/locate/entcs

Supermetrics over Apartness Lattice-Ordered Semigroup [1]

Gabriel Ciobanu [2]

"A.I.Cuza" University, Faculty of Computer Science
700506 Iaşi, Romania

Dănuţ Rusu [3]

"A.I.Cuza" University, Faculty of Mathematics
700506 Iaşi, Romania

Abstract

We define the notion of supermetrics over apartness lattice-ordered semigroups, present the relationship between supermetrics and additive functions, and prove several conditions of metric completeness.

Keywords: set-set apartness, lattice-ordered semigroup, metrics, completion.

1 Introduction

According to the set-set apartness relation introduced in [12], A and B are apart if their intersection is an empty set. We considering a weaker form of (set-set) apartness where two sets A and B have common elements, but these elements are not considered when we define the apartness relation between A and B.

Let $\mathcal{C}$ be a family of subsets of a nonempty set X satisfying the following axioms:

$$A \cup B \in \mathcal{C}, \text{ for all } A, B \in \mathcal{C};$$
$$A \backslash B \in \mathcal{C}, \text{ for all } A, B \in \mathcal{C}.$$

From these axioms it follows that $A \cap B \in \mathcal{C}$, and $A \triangle B \in \mathcal{C}$ for all $A, B \in \mathcal{C}$; moreover $\emptyset \in \mathcal{C}$. Therefore $(\mathcal{C}, \triangle, \emptyset)$ is a monoid, and $(\mathcal{C}, \cup, \cap, \emptyset)$ is a lattice with the least element $\emptyset$. Additionally, we have $A \triangle B = (A \cup B) \triangle (A \cap B)$ for all $A, B \in \mathcal{C}$.

[1] Research partially supported by CEEX Grant no.133/2006

[2] Email: gabriel@info.uaic.ro

[3] Email: drusu@uaic.ro

doi:10.1016/j.entcs.2008.03.015

These facts inspire us to define a new structure called apartness lattice-ordered semigroup $(S, \cdot, \vee, \wedge, u)$. In such a structure we impose the distributivity of $\cdot$ with respect to $\vee$ and $\wedge$. The distributivity properties are not satisfied by the above example where we only have $A\triangle(B \cup C) \subseteq (A\triangle B) \cup (A\triangle C)$ and $A\triangle(B \cap C) \supseteq (A\triangle B) \cap (A\triangle C)$. However we can find such properties in the structure of natural number set and in divisibility theory, to mention only few of the examples presented in Section 2.

Let $\mu : \mathcal{C} \to \mathbb{R}_+$ be an additive function on $\mathcal{C}$. Then the function $d : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$ defined by $d(A, B) = \mu(A\triangle B)$ is a pseudo-metric on $\mathcal{C}$. μ is a metric whenever $\mu(A) = 0$ iff $A = \emptyset$. Since μ is an additive function, then it is (increasing) monotone. Moreover, we have $d(A, B) = \mu(A \cup B) - \mu(A \cap B)$. Considering the structure of apartness lattice-ordered semigroup $(S, \cdot, \vee, \wedge, u)$ and an increasing monotone function $f : S \to \mathbb{R}$, it is possible to define a function $d : S \times S \to \mathbb{R}_+$ by $d(x, y) = f(x \vee y) - f(x \wedge y)$. This function d has several nice properties, and we call it a supermetric. We study some of these properties, including several conditions related to metric completeness.

The structure of the paper is as follows. Section 2 presents the apartness lattice-ordered semigroups. Section 3 defines the supermetrics over this structure, and it provides the main novel concept of the paper. In Section 4 we present the relationship between supermetrics and additive functions. Some results related to the complete apartness lattice-ordered semigroups and to their completions are presented in Sections 5 and 6.

2 Apartness lattice-ordered semigroups

Definition 2.1 An apartness lattice-ordered semigroup (alo-semigroup) is a particular lattice-ordered semigroup system $(S, \cdot, \vee, \wedge, u)$, where

1. $(S, \cdot, u)$ is a semigroup with unit u (monoid);
2. $(S, \vee, \wedge, u)$ is a lattice with the least element u; if S has a zero z, $z \neq u$ and z is the greatest element;
3. for every $a, b, c \in S$ we have

$$\text{(i)} \quad ab = (a \vee b)(a \wedge b)$$
$$\text{(ii)} \quad a(b \vee c) = ab \vee ac$$
$$\text{(iii)} \quad a(b \wedge c) = ab \wedge ac$$

The usual order in an alo-semigroup $(S, \cdot, \vee, \wedge, u)$ is the order induced by its lattice:

$$x \leq y \text{ iff } x = x \wedge y.$$

Let S be an alo-semigroup and $R \subseteq S$. R is a sub alo-semigroup of S if R is an alo-semigroup with respect to the operations of S restricted to R, and if R (as a lattice) is a sublattice of S.

We give some examples of alo-semigroups:

1. $S = \{f \mid f : [0, 1] \to [0, 1]\}$, where $(f \wedge g)(x) = max(f(x), g(x))$, $(f \vee g)(x) =$