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Normalization of N-Graphs via Sub-N-Graphs

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Abstract

Alves presented in his PhD thesis a normalization procedure for N-Graphs, a multiple conclusion natural deduction for propositional classical logic proposed by de Oliveira in 2001, with proofs as directed graphs. Here we develop a new normalization for N-Graphs inspired by A. Carbones work in 1999, where she proposed a combinatorial model to study the evolution of proofs during the procedure of cut elimination.

Keywords: N-Graphs, Normalization, Directed graphs, Duplication

1 Introduction

Whenever one is concerned with the study of proofs from a geometric perspective one can hardly overestimate the pioneering work of Statman in his doctoral thesis Structural Complexity of Proofs [19]. Drawing on Statman's legacy, for the last three decades at least two research programmes have approached the study of structural properties of formal proofs from a geometric perspective: (i) the notion of proof-net, given by Girard in [12] in the context of linear logic; and (ii) the notion of logical flow graph given by Buss in [6] and used as a tool for studying the exponential blow up of proof sizes caused by the cut-elimination process, in this case giving rise to a programme (1996–2000) proposed by Carbone in collaboration with Semmes [7]. Statman's geometric perspective has given an important legacy,

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namely the idea of extracting structural properties of proofs in natural deduction (ND) using appropriate geometric tools and intuitions. The lack of symmetry in ND presents a challenge for such a kind of study. Of course, the obvious alternative is to look at multiple-conclusion calculi. One can find in the literature different approaches involving such calculi, such as, for example, Kneale's tables of development [13] (studied in depth by Shoesmith & Smiley [18]) and Ungar's multiple-conclusion ND [20]. But then a great challenge remained: normal forms and the normalization procedure.

The system of N-Graphs, a multiple conclusion ND for propositional classical logic developed in the early 2000's by de Oliveira [14] out of a combination of the techniques developed in the two aforementioned research programmes, has revealed itself as a rather appropriate framework in which to formulate and explore techniques for normalizing ND proofs in the form of directed graphs. N-Graphs were motivated by the idea of proofs as geometric objects and aimed towards the study of the geometry of Natural Deduction systems. Following that line of research, we propose a normalization procedure defined as a set of combinatorial operations on graphs that can offer a framework for future combinatorial studies on the proof growth during normalization. The procedure we present in this paper also works as an extension of the normalization defined by Prawitz, i.e. it enjoys the separation and subformula properties.

In her analysis of the blow-up of proof-size after cut elimination in sequent calculus proofs, Carbone defined an operation called duplication and worked with the logical flow graphs extracted from sequent calculus proofs in order to propose a purely combinatorial analysis of cut elimination [8]. Our procedure uses Alves' original beta and permutative reductions [2], changing reductions with the link with meta-edge. A new set of switchable reductions is presented, combined with an adaptation of the duplication operation for sub-N-Graphs, to handle switchable links. As a result, this new normalization has a stronger parallel with both Prawitz's normalization and Gentzen's cut-elimination, offering a good start point for studies on correspondences between those two procedures, like the ones presented by Zucker [21] and Ungar [20]. This also offers a base to extend Carbone's results on proof growth during cut-elimination in sequents to a ND system.

2 N-Graphs

Proposed by de Oliveira [14,15], N-Graphs is a symmetric natural deduction (ND) calculus with the presence of structural rules, similar to the sequent calculus. It is a multiple conclusion proof system for classical logic where proofs are built in the form of directed graphs (*"digraphs"*). Several studies have been developed on N-Graphs since its first publication in 2001 [14], like Alves' development on the geometric perspective and cycle treatment towards the normalization of the system [3] and Cruz's definition of intuitionistic N-Graphs [9]. A normalization algorithm was presented for classical N-Graphs [1], along with the subformula and separation properties [2]. Also, a linear time proof checking algorithm was proposed [4], and