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Electronic Notes in Theoretical Computer Science 323 (2016) 143–161

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

# Fibrational Modal Type Theory

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**Abstract**

This paper describes a fibrational categorical semantics for the modal necessity-only fragment of constructive modal type theory, both with and without dependent types. Constructive type theory does not usually discuss logical modalities, and modalities tend to be mostly studied within classical logic, not within type theory. But modalities should be very useful in type theory, as they are very useful in modelling theoretical computing systems. Providing constructive versions of modal logics and their associated Curry-Howard modal type theories is also a very productive program, e.g. helpful when dealing with computational effects, staged computation, and functional reactive types, for example. There seems to be renewed interest in the notion of constructive modal type theory (and in notions of linear type theory), in part because of the interest in homotopy type theory. The modal type theory presented here uses dependent types, in the style of Ritter's categorical models of the Calculus of Constructions. To build up to these, we first discuss the kinds of constructive modal type theory in the literature. Then we provide a non-dependent modal type theory, introduced in previous work, that we generalize to dependent types in the following section. Dependent type theories are usually but not always given categorical semantics in terms of *fibrations*. We provide semantics in terms of fibrations for both the non-dependent and the dependent type systems discussed and prove them sound and complete, thereby providing evidence that the type theory is meaningful. These fibrational models should be also applicable to the homotopy type theory setting.

*Keywords:* modal logic, fibrations, categorical models

## 1 Introduction

Modal logic is the formal logic system that extends propositional (or predicate) logic to include operators expressing modality, mostly intensional notions of possibility and necessity, but also temporal, deontic, provability and other kinds of modalities.

[1] We would like to thank the editors Mário Benevides and René Thiemann for their patience beyond the call of duty with typesetting issues.

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http://dx.doi.org/10.1016/j.entcs.2016.06.010
1571-0661/

Modal logic, originally conceived as the logic of necessity and possibility, has evolved into a mathematical discipline of its own that deals with (restricted) description languages for talking about various kinds of relational structures.

Modal logic is the explicit logical system most used in Computer Science, but almost all of its uses are based on *classical* modal logic, that is on modal logic over a classical propositional logic basis, while in this paper we deal with constructive modal logic. Constructive (or intuitionistic, we will use the terms interchangebly) modal logic starts from an intuitionistic propositional basis and add modalities to it. This process, like any of 'constructivizing' a logical system, is usually a one-to-many one, with a single classical concept giving rise to many possible constructive versions, which one needs to compare, contrast and choose from. All kinds of mathematical, philosophical and esthetical criteria can be used for choosing your favorite constructive version of a classical concept.

The basic unary (1-place) modal operators are usually written $\Box$ for *necessarily the case* and $\Diamond$ for it is *possibly the case*. In a classical modal logic, each of these operators can be expressed using the other using negation: $\Diamond P \leftrightarrow \neg\Box\neg P$; $\Box P \leftrightarrow \neg\Diamond\neg P$. In the case of constructive modal logics this interdefinability of the operators is not expected nor desired. The same way as we do not have in intuitionistic logic the interdefinability between universal and existential quantifiers $\forall x.P(x) \leftrightarrow \neg\exists x\neg P(x)$, $\exists x.P(x) \leftrightarrow \neg\forall x\neg P(x)$ we do not expect it as far as the modalities are concerned.

Some times different methods of constructivization in logic led researchers to the same systems. In particular the work of Fitch [18], Fisher-Servi [17], Ewald [15], Plotkin and Stirling and especially Simpson [29] has resulted in the most well-known and successful systems of intuitionistic modal logic, based on the system $\mathsf{IK}$. Simpson's systems use a labelled deduction method, where the semantic intuitions of possible worlds are codified in the syntax, via assertions about accessibility between worlds. Following a very different path Prawitz also provided intuitionistic versions for systems S4 and S5 in his seminal work in Natural Deduction [26]. This work gave rise to a collection of associated work, especially on intuitionistic versions of S4, [8,7,16,2], where the guiding intuitions come from the Curry-Howard correspondence. These systems are here dubbed the systems CS4, for constructive modal logic.

Some of the research on these constructive modal systems based on Prawitz-style Natural Deduction were inspired by Girard's Linear Logic and these benefit from later developments in linear lambda calculi, such as the Dual Intuitionistic and Linear lambda-calculus (DILL) [4] of Barber and Plotkin. Our previous work on constructive modal logic provides a Dual Intuitionistic Modal lambda-calculus (DIML) [19] in the style of DILL. Since the work on DIML was basically motivated by the implementation of functional languages using categorical combinators, the main work on the DIML calculus was to make sure that the *explicit substitutions* necessary for the implementation of the categorical combinators worked. But we also presented the basics of the constructive modal type theory for necessity-only S4, which we reproduce in the next section.