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Electronic Notes in Theoretical Computer Science 323 (2016) 215–233

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

On Graphs for Intuitionistic Modal Logics [1]

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Abstract

We present a graph approach to intuitionistic modal logics, which provides uniform formalisms for expressing, analysing and comparing Kripke-like semantics. This approach uses the flexibility of graph calculi to express directly and intuitively possible-world semantics for intuitionistic modal logics. We illustrate the benefits of these ideas by applying them to some familiar cases of intuitionistic multi-modal semantics.

Keywords: Intuitionistic modal logics, semantics, graph formulations, calculi, refutation, special relations.

1 Introduction

We present a graph approach to intuitionistic modal logics, which provides a flexible and uniform tool for expressing, analysing and comparing possible-world semantics.

This graph approach can be regarded as a version of diagrammatic reasoning, where we can express formulas by diagrams, which can be manipulated to unveil properties (like consequence and satisfiability). Graph representations and transformations, having precise syntax and semantics, give proof methods. An interesting feature of this graph approach is its 2-dimensional notation providing pictorial representations that support visual manipulations [4]. These ideas have been adapted to refutational reasoning [14] and applied to multi-modal classical logics [15].

[1] Research partly sponsored by the Brazilian agency CNPq. Helpful comments by the referees are gratefully acknowledged.

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http://dx.doi.org/10.1016/j.entcs.2016.06.014
1571-0661/

Modal logics and graphs are closely connected. Kripke semantics can be presented via directed labelled graphs for the accessibility relation of each modality [2]. It is natural to represent that a is related to b via relation r by an arrow $a \xrightarrow{r} b$.

Intuitionistic modal logic is an interesting subject [6,11]. There seems to be little consensus on the appropriate approach to its semantics, as indicated by the diversity of Kripke-like semantics proposed (see [5,13] and references therein).

We provide graph calculi, having diagrams as terms and whose rules transform diagrams, capturing graphically the semantics of the modal operators and accessibility relations. These calculi provide uniform and flexible formalisms where one can explore Kripe-like semantics for intuitionistic modal logics: satisfaction conditions, valid formulas, etc. We illustrate these ideas by 2 case studies: logics as in [13,5].

We will consider a modal language ML, with set Φ of formulas, given by sets $\mathbb{PL}$, of propositional letters, and $\mathbb{RS}$, of 2-ary relation symbols. The formulas of ML are generated by the grammar $\varphi \coloneqq \bot \mid \mathrm{p} \mid \varphi' \wedge \varphi'' \mid \varphi' \vee \varphi'' \mid \varphi' \to \varphi'' \mid \langle \mathrm{r} \rangle \varphi \mid [\mathrm{r}]\varphi$. [4]

2 Graphs and Modalities: Basic Ideas

We now introduce informally some basic ideas about graphs and modalities. [5]

A graph amounts to a finite set of (alternative) slices. A slice S consists of an underlying draft $\underline{\mathrm{S}}$ together with a distinguished node (marked, e. g. $\widehat{\mathrm{w}}$). A draft amounts to finite sets of nodes and arcs. Slices and graphs represent sets of states, whereas drafts (and sketches, see Section 3) will describe restrictions on states.

Arcs may be binary or unary. A binary arc stands for accessibility between states; we represent that node v is accessible from node u by the relation of r by a solid arrow labelled r from u to v: $\mathrm{u} \xrightarrow{\mathrm{r}} \mathrm{v}$ (abbreviated $\mathrm{u\,r\,v}$). A unary arc is meant to capture the fact that a formula holds at a state; we represent that formula φ holds at node w by a dashed line from w to φ: $\mathrm{w} \dashrightarrow \!\!\prec \varphi$ (abbreviated $\mathrm{w}|\varphi$).

Expressions will encompass slices, graphs and their complements (noted by an overbar). As such, an expression represents a set of states; so we can also use unary arcs of the form $\mathrm{w} \dashrightarrow\!\!\prec \mathrm{E}$, where E is an expression.

We now introduce some concepts to be used and illustrated in Example 2.1.

A (draft) morphism is a node mapping that preserves arcs. A (slice) homomorphism is a morphism of their underlying drafts that preserves distinguished nodes. A draft may have conflicts that prevent its satisfaction. We consider two kinds of conflicts. One concerns contradictory 1-ary arcs: if draft D has the pattern $\mathrm{E} \succ\!\!\dashleftarrow \mathrm{w} \dashrightarrow\!\!\prec \overline{\mathrm{E}}$, then expression E is a witness of a conflict at node w. If D has 1-ary arc $\mathrm{w} \dashrightarrow\!\!\prec \overline{\mathrm{Q}}$, slice Q will be a witness of a conflict at node w if there is a morphism from $\underline{\mathrm{Q}}$ to D mapping the distinguished node of Q to w.

To reason about modal formulas, we convert them to expressions (with the same meaning) and reason graphically about these. We reduce consequence to unsatisfiablility: "every state satisfying $\psi_1, \ldots, \psi_n$ also satisfies θ" (noted $\{\psi_1, \ldots, \psi_n\} \models$

[4] As usual, $\neg\varphi$ abbreviates $\varphi \to \bot$.

[5] These and other ideas will be formulated more precisely later on: in Section 3.