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A Short Introduction to Clones

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Abstract

In universal algebra, clones are used to study algebras abstracted from their signature. The aim of this paper is to give a brief introduction to the theory thereof. We give basic definitions and examples, and we present several results and open problems, selected from almost one hundred years of ongoing research. We also discuss what is arguably the most important tool to study clones – the Galois connection between operations and relations built on the notion of preservation. We conclude the paper by explaining the connection between clones and the closely related category theoretic notion of Lawvere theory.

Keywords: clones, relational clones, composition closed classes, polymorphisms, invariant relations

1 Introduction

A set of functions is a clone if and only if it is the set of non-nullary term functions of some algebra. For this reason, clones can be thought of as a representation of algebras which abstracts from their signature.

This paper aims at being a brief introduction to their theory. After discussing the basic definitions, we present some of the most celebrated results from almost one hundred years of ongoing research in the field. Although there are far more significant results than we can mention in this short survey, there are even more open problems. Indeed, as soon as the cardinality of the set A exceeds two, very little is known about the structure of the lattice of all clones on A. We will present some of the most outstanding open problems, giving the reader an impression of

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what we know and don't know about the seemingly incomprehensible variety of clones.

To study clones, people have used many different techniques, coming from fields such as combinatorics, set theory or topology. One technique, however, stands above all others and is arguably the most important item in a clone theorist's toolbox. It is the Galois connection Pol-Inv between operations and relations based on the notion of preservation. This connection, nicknamed the "most basic Galois connection in algebra" in [27], has literally been used thousands of times and is the heart and soul of many important contributions to the theory of clones. We explain this technique, illustrate its usefulness and point to some of its many specifications and generalizations.

As clones are composition-closed sets of functions that contain projections, it is perhaps not surprising that the concept was generalized to category theory. In his 1963 PhD thesis [22], Bill Lawvere introduced the notion of algebraic theory (nowadays called Lawvere theory), which can be thought of as a category theoretic abstraction of clones. Shortly after Lawvere's thesis was published, clone theorists captured the same level of abstraction in the notion of abstract clone [7,44,46]. We present these notions and explain their connection, also discussing how concrete questions from classical clone theory (even if asked for finite base sets) benefit from a more abstract view.

The paper is structured as follows: Following the introductory words, we start our short survey in Section 2, which contains basic definitions and motivating examples. We continue in Section 3 by giving some examples of typical objects of research in the field, including a selection of results and open problems. The fourth section explains the Galois connection Pol-Inv, and the last section presents the connection between clones and Lawvere theories.

It is important to note that this survey is just a brief introduction and contains only some (sometimes almost randomly chosen) examples of the research that has been going on for several decades. For a complete overview of the theory we refer to the monographs [32,43,21].

2 What is a clone?

Given a universal algebra (A, F), where F is a set of finitary operation on the set A, one is often interested in the term functions of the algebra rather than in the set F itself. In particular, if two algebras have the same set of term functions, then one might consider their difference as a mere question of representation. This motivates a notion that describes precisely those sets of functions that can arise as sets of term functions of an algebra – and that is exactly what a clone is:

Definition 2.1 Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, \ldots\}$. For $n \in \mathbb{N}$, denote by $O_A^{(n)}$ the set of n-ary operations on A and set $O_A := \bigcup_{n\in\mathbb{N}_+} O_A^{(n)}$. A subset $C \subseteq O_A$ is a called a *clone* (or *clone of operations*) if it contains all the projection mappings $\mathrm{pr}_i^k\colon A^k \to A : (x_1, \ldots, x_k) \mapsto x_i$ and is closed with respect to superposition of

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