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Freyd categories are Enriched Lawvere Theories

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Abstract

Lawvere theories provide a categorical formulation of the algebraic theories from universal algebra. Freyd categories are categorical models of first-order effectful programming languages.

The notion of *sound limit doctrine* has been used to classify accessible categories. We provide a definition of Lawvere theory that is enriched in a closed category that is locally presentable with respect to a sound limit doctrine.

For the doctrine of finite limits, we recover Power's enriched Lawvere theories. For the empty limit doctrine, our Lawvere theories are Freyd categories, and for the doctrine of finite products, our Lawvere theories are distributive Freyd categories. In this sense, computational effects are algebraic.

Keywords: Freyd categories, Lawvere theories, monads and notions of computation.

1 Introduction

Strong monads have helped to organize the semantics of impure programming languages from at least two perspectives: firstly by examining the crucial properties of concrete models of programming languages; secondly by axiomatizing the equations between programs that must hold in all models [24,25].

However, more refined perspectives have since emerged.

- Firstly, the monads involved in many concrete models of impure programming languages actually arise as free algebras for equational theories, in the setting of enriched category theory (e.g. [28,29]).
- Secondly, when we separate first-order effectful computation from higher-order types, we arrive at the notion of Freyd categories as an axiomatization of first-order effectful computation. (Moggi's monad-models can be recovered as closed Freyd categories, see e.g. [19].)

In this paper I explain that the second development can be seen as an instance of the former.

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Informal overview

The generalization from traditional equational theories to enriched ones proceeds as follows. Recall that in a traditional algebraic signature there is a set of n-ary operations for each natural number n, and so a structure for the signature comprises a function $X^n \to X$ for each n-ary operation.

This is enriched by replacing the category of sets by a category $\mathcal{V}$ (perhaps the simplest interesting example of $\mathcal{V}$ to have in mind is the category of posets and monotone functions). The n-ary operations are no longer required to form a set, but rather an object O_n of $\mathcal{V}$; the arities n are no longer natural numbers, but rather 'finitary' objects of $\mathcal{V}$; and a structure for such a signature in a given $\mathcal{V}$-enriched category $\mathcal{A}$ comprises a morphism $O_n \to \mathcal{A}(X^n, X)$ in $\mathcal{V}$, where X^n is a power. (This line of thought goes back to Kelly's work [14,15]; our starting point is Power's development [30]; see also [34] for an overview.)

A traditional equational theory determines a Lawvere theory, which is a category where the objects are natural numbers, and a morphism $m \to 1$ is a term in m variables modulo the equations, and in general a morphism $m \to n$ is a family of n terms-mod-equations in m variables. The categories arising in this way can be characterized as categories $\mathbb{L}$ with finite products equipped with a functor $J : \mathbb{N}^{\mathrm{op}} \to \mathbb{L}$, where $\mathbb{N}$ is the category of natural numbers and functions between them; the functor J is required to be identity-on-objects (i.e. $\mathbb{N}^{\mathrm{op}}$ and $\mathbb{L}$ have the same objects) and to preserve products.

Similarly a $\mathcal{V}$-enriched Lawvere theory [30] is defined to be a $\mathcal{V}$-enriched category $\mathbb{L}$ with 'finitary' powers and an identity-on-objects finitary-power-preserving $\mathcal{V}$-functor $\mathbb{F}^{\mathrm{op}} \to \mathbb{L}$, where $\mathbb{F}$ is the category of finitary objects of $\mathcal{V}$.

On the other hand, the notion of Freyd category arose in the work of Levy, Power and Thielecke [19,33] as a categorical framework for first order effectful programs. Recall the basic ideas of the categorical interpretation of type theory: that types are denoted by objects of a category, that a context is denoted by the product of its constituent types, and that a judgement $\Gamma \vdash t : \tau$ is interpreted as a morphism $\Gamma \to \tau$ in the category. A Freyd category comprises two categories with the same objects: one $\mathbb{V}$, whose morphisms denote pure, value judgements, and one $\mathbb{C}$, whose morphisms denote judgements of computations, together with an identity-on-objects functor $J : \mathbb{V} \to \mathbb{C}$. For example, $\mathbb{C}$ might be the Kleisli category for a strong monad on $\mathbb{V}$. Since the order of effectful computation matters, $\mathbb{C}$ typically does not have products, but it does have a product-like structure, and the functor J is required to preserve it. This was initially described in terms of premonoidal categories [32]. Subsequently, Levy used a formulation based on actions of monoidal categories [18, App. B] (see also [23]) and that is what we use in this paper.

Coming back to the definition of enriched Lawvere theory, notice that, naively put, there is some choice in what is meant by 'finitary' when it comes to the arities. When $\mathcal{V} = \mathbf{Set}$, 'finitary' means finite. Power takes $\mathcal{V}$ to be a locally finitely presentable category, and 'finitary' means finitely presentable. If $\mathbb{V}$ is a category with finite products, and $\mathcal{V}$ is the functor category $[\mathbb{V}^{\mathrm{op}}, \mathbf{Set}]$, then we can take 'finitary' to mean representable. In this case, an enriched Lawvere theory is the