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Characterising E-projectives via Co-monads

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Abstract

This paper demonstrates the usefulness of a comonadic approach to give previously unknown characterisation of projective objects in certain categories over particular subclasses of epimorphisms. This approach is a simple adaptation of a powerful technique due to M. Escardó which has been used extensively to characterise injective spaces and locales over various kinds of embeddings, but never previously for projective structures. Using some examples, we advertise the versatility of this approach – in particular, highlighting its advantage over existing methods on characterisation of projectives, which is that the comonadic machinery forces upon us the structural properties of projectives *without* relying on extraneous characterisations of the underlying object of the co-algebra arising from the comonad.

Keywords: E-projectives, KZ-comonads, Right U-quotients, Ordered monoids, Normal semi-rings, Semilattices, Z-frames

1 Introduction

The problem of characterizing projectives and their duals, injectives, in various categories has a long history in mathematics with its origin tracing back to module theory, e.g., characterizing projective modules in module theory. An object P of a category $\mathbf{C}$ is *projective* if for every epimorphism $e : A \longrightarrow B$ and every morphism $f : P \longrightarrow B$, there is a $\mathbf{C}$-morphism (not necessarily unique) $f' : P \longrightarrow A$ such that $f = f' \circ e$. In the category of sets, every object is projective, while the only projectives in the category of groups are the free ones. In algebra, projective objects are viewed as a generalization of free objects.

Much later on, attention with regards to the study of projectives and injectives was shifted from algebraic structures to ordered structures and topological spaces. R. Sikorski showed that the injectives in the category of boolean algebras are precisely the complete boolean algebras ([19]). Later, R. Balbes extended this result to

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show that the injectives in the category of distributive lattices are also the complete boolean algebras ([2]). G. D. Crown in [5] characterised the projectives (and injectives) in the category of sup complete lattices as the completely distributive lattices. Another important example is D.S. Scott's discovery that the injective T_0-spaces are exactly the continuous lattices with their Scott-topology [18]. Notably, in their effort of generalizing Scott's result to frames, B. Banaschewski and S. B. Niefeld in [4] reported that the only projective frame is **2** the two-element chain. Indeed, a similar result has been already been reported in [3] for the category of distributive lattices. For situations like these where projectives are scarce, it is natural to consider more general types of projective objects, whence the notion of **E**-projectives. To achieve this, one takes not all the epimorphisms but only a certain subclass **E** of epimorphisms. More precisely, an object P of a category **C** is **E**-projective or projective over the **E**-morphisms if for every **C**-morphism $f : P \longrightarrow A$ and every **E**-morphism $e : A \longrightarrow B$, there is a **C**-morphism $f' : P \longrightarrow A$ such that $f = f'e$. Dually, there is the notion of **E**-injectives.

In [4] B. Banaschewski and S. B. Niefeld showed that the projectives over **E** the collection of *regular epimorphisms* (i.e., morphisms which are co-equalisers of some parallel pair of morphisms) in the category of frames are exactly the stably completely distributive lattices; thereby relaxing on the condition of projectivity to admit a larger class of objects. In the literature, projectives over the regular epimorphisms are also called *regular projectives*. In certain categories such as the category **KHausSp** of compact Hausdorff spaces and continuous maps, every epimorphism is regular. Then, for such categories, the problem of characterising projectives then amounts to that of characterising the regular projectives; and this latter problem, in certain situations, may turn out to be easier. One such instance is Gleason's theorem, i.e., the projective objects in **KHausSp** (which coincide with the regular projective objects) are precisely the extremally disconnected spaces (refer to the proof found in [13, pp. 98–103]).

Recent years have seen a continued interest in the both the areas of characterisation of the **E**-injectives and projectives for poset-enriched categories. Along the 'injective' line, one important advancement was made by M. H. Escardó. His technique involves the use of KZ-monads to characterise injectives over certain embeddings in some poset-enriched categories. Escardó's method relies heavily on the following result:

Theorem 1.1 *([9, Theorem 4.2.2, p.32])*
Let T be a KZ-monad on $\mathbf{X}$. Then, the following are equivalent for any $A \in \mathbf{X}$:

(i) A is right injective over right T-embeddings.

(ii) A is injective over right T-embeddings.

(iii) A is a T-algebra.

This method is particularly powerful since this monadic approach characterises the injectives over right T-embeddings (where T is the given KZ-monad on the category) as precisely the underlying algebras of the monad T. The KZ-monadic machinery then forces upon us the characterisation of these underlying algebras by