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Generalized Scott Topology on Sets with Families of Pre-orders

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Abstract

[5, Lei Fan] proposed a class of sets with families of pre-orders ($\mathcal{R}$-posets for short). They are not only a non-symmetric generalization of sfe [8, L.Monteiro] but also a special case of quasi-metric spaces (qms, [10, M. B. Smyth]) and generalized ultrametric spaces (gums, [9, J. J. M. M. Rutten]). In this paper, we define a kind of generalized Scott topology on $\mathcal{R}$-posets and discuss some basic properties of the topology. Some relevant interesting examples are offered. It is worth pointing out that an $\mathcal{R}$-monotone functions is $\mathcal{R}$-continuous if and only if (iff for short) it's continuous with respect to (w.r.t for short) the generalized Scott topology.

Keywords: $\mathcal{R}$-posets, Scott Topology, Generalized Scott Topology, $\mathcal{R}$-continuous.
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1 Introduction and Preliminaries

Domain theory is an important branch of computer science, motivated by providing a mathematical foundation of computer functional languages. Information orderings is proposed and applied to interpret quantity or extent of approximating to

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some object. Since information need not to be always comparable, the information ordering is a partial order.

Definition 1.1 A partial order is a binary relation $\sqsubseteq$ over a set P which is antisymmetric, transitive, and reflexive. In other words, a partial order is an antisymmetric preorder. A set with a partial order is called a partially ordered set (also called a poset).

As a supplement and enrichment of partial order, the second author of this work proposed a structure of sets with families of pre-orders in [5]. Inside this mathematical structure, there will be a possibility to interpret or compare information yielded from complex computation process.

Definition 1.2 [[5], L. Fan] Let $(P, \sqsubseteq)$ be a poset and let $(\omega, \leq)$ be the set of natural numbers. If $\mathcal{R} = (\sqsubseteq_n)_{n\in\omega}$ is a family of pre-orders on P, where $\sqsubseteq_0 = P \times P$, such that (i)$\forall n, m \in \omega, m \leq n$ implies $\sqsubseteq_n \subseteq \sqsubseteq_m$, and (ii)$\bigcap_{n\in\omega} \sqsubseteq_n = \sqsubseteq$, then we call $(P, \sqsubseteq)$ a poset with the pre-order family $\mathcal{R}$. We call it $\mathcal{R}$-poset or rpos for short and denote it briefly by $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ or $(P, \sqsubseteq; \mathcal{R})$. $(P, \sqsubseteq)$ is said to be a trivial rpos when $\mathcal{R} = (\sqsubseteq)_{n\in\omega}$.

J.P Gao simulated state transition systems by the notion of rpos in [6] and gave a general method to obtain rpos from sets.

Example 1.3 [[6,8,9]] Consider a transition system $\langle S, A, \to\rangle$. For $a \in A$ and $U \subseteq S$, let $p_a U = \{s | (\exists t \in U) s \xrightarrow{a} t\}$ be the set of a-predecessors of U. Extend this to traces by $p_\varepsilon U = U$ and $p_{av} = p_a p_v U$. Then $s \in p_v S$ if and only if s has trace v. Let $\mathcal{U}_n = \{p_v S | v \text{ has length } n\}$, so that $\mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n = \{p_v S | v \text{ has length } \leq n\}$. We define $s \sqsubseteq_n t$ if and only if, for every trace v of length $\leq n$, $s \in p_v S$ implies $t \in p_v S$, that is, if s has traces of length $\leq n$ then t has the same traces of length $\leq n$. Then $(\sqsubseteq_n)_{n\in\omega}$ is a pre-order family on S.

Example 1.4 [[6,8]]Let S be a set and $(\mathcal{U}_n)_{n\in\omega}$ a family of sets $\mathcal{U}_n$ of subsets of S where $\mathcal{U}_0 = \{S\}$. Define $s \sqsubseteq_n t$ by requiring that $s \in U$ implies $t \in U$ for every $U \in \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n$. This gives an rpos.

Example 1.5 We give several interesting examples of rpos in Figures A.1–A.6 in Appendix A. In each figure, the orders in $(\sqsubseteq_n)_{n\in\omega}$ and $\sqsubseteq = \bigcap_{n\in\omega} \sqsubseteq_n$ are represented as Hasse diagrams. Throughout the paper we always assume that the pre-orders in each figure are in turn $\sqsubseteq_1, \sqsubseteq_2, \sqsubseteq_3, \cdots$, and $\sqsubseteq = \bigcap_{n\in\omega} \sqsubseteq_n$ unless otherwise noted.

For a poset $(P, \sqsubseteq)$ with a pre-order family $\mathcal{R} = (\sqsubseteq_n)_{n\in\omega}$, it is easy to see that $\sqsubseteq \subseteq \sqsubseteq_n$ for all $n \in \omega$, thus pre-order family $(\sqsubseteq_n)_{n\in\omega}$ can be seen as a simplicity sequence of $\sqsubseteq$ on P. We interpret $x \sqsubseteq_n y$ as indicating the extent $(1/n)_{n\in\omega}$ (the smaller the better) to which the transitions of x can be simulated by y. Thus, it is not surprising that we assume that any two elements are in relation $\sqsubseteq_0$ and that $\sqsubseteq_n \subseteq \sqsubseteq_m$ for $m \leq n$ in ω. A typical application of this notion is to objects that can be structured or evaluated in stepwise manner, where it makes sense to state that an object can be simulated by another object up to level n.