Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science 301 (2014) 179–188

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

# Meet Precontinuous Posets

Wenfeng Zhang[1]

*Department of Mathematics, Sichuan University, Chendu, China.*

Xiaoquan Xu[⋆,2]

*Department of Mathematics, Jiangxi Normal University, Nanchang, China.*

**Abstract**

In this paper, we introduce the concept of meet precontinuous posets, a generalization of meet continuous lattices to posets. The main results are: (1) A poset $P$ is meet precontinuous iff its normal completion is a meet continuous lattice iff a certain system $\gamma(P)$ which is, in the case of complete lattices, the lattice of all Scott-closed sets is a complete Heyting algebra; (2) A poset $P$ is precontinuous iff the way below relation is the smallest approximating auxiliary relation iff $P$ is meet precontinuous and there is a smallest approximating auxiliary relation on $P$. Finally, given a poset $P$ and an auxiliary relation on $P$, we characterize those join-dense subsets of $P$ whose way-below relation agrees with the given auxiliary relation.

*Keywords:* Precontinuous poset, meet precontinuous poset, meet continuous lattice, normal completion, auxiliary relation.

## 1 Introduction

Domain theory was introduced by Scott in the late 60s for the denotational semantics of programming languages. It provides the mathematical foundation for the design, definition, and implementation of programming languages, and for systems for the specification and verification of programs. From both the computer science side and the purely mathematical side, one of important aspects of domain theory is to carry as much as possible of the theory of continuous domains to as general an ordered structure as possible. Due to their strong connections to computer science, general topology and topological algebra, continuous domains have been

⋆ Supported by the National Natural Science Foundation of China (Nos. 10861007, 11161023), the Fund for the Author of National Excellent Doctoral Dissertation of China (No. 2007B14), the NFS of Jiangxi Province (Nos. 20114BAB201008), the Fund of Education Department of Jiangxi Province (No. GJJ12657).

[1] Email: zhangwenfeng2100@163.com

[2] Email: xiqxu2002@163.com

http://dx.doi.org/10.1016/j.entcs.2014.01.015

extensively studied by people coming from various areas [1,7]. There are several different equivalent ways to define continuous domains, the most straightforward one is formulated by using the way below relation. We say $x$ *way below* $y$, denoted by $x \ll y$, iff $x \in \Downarrow y := \bigcap\{\downarrow D : D$ is directed and $y \leq \vee D\}$. A dcpo $P$ is called a *continuous domain* if each of the sets $\Downarrow y$ is directed and has join $y$. This definition has turned out to be very fruitful for many categorical and topological developments generalizing the theory of continuous lattices, but it is still rather restrictive, taking into consideration only dcpos. So, there are more and more occasions to study posets which miss suprema of directed sets (see [5,10,12,15,18]). Though continuous posets inherit some good properties of continuous domains, they also fail to satisfy some hopeful properties, for example, they are not *completion-invariant* [3], i.e. the normal completion of a continuous poset is not always a continuous lattice (see [2]). In [2], Erné introduced a new way below relation and the concept of precontinuous posets by taking Frink ideals [6] instead of directed lower sets, which are not restricted to dcpos and has the desired completion-invariant property.

A meet continuous lattice is a complete lattice in which binary meets distribute over directed suprema (see [5]). This algebraic notion has a purely topological characterization that can be generalized to the setting of directed complete partial orders (dcpos) in [7,9]: A dcpo $P$ is *meet continuous* if for any $x \in P$ and any directed subset $D$ with $x \leq \sup D$, one has $x \in cl_{\sigma(P)}(\downarrow x \cap \downarrow D)$, where $cl_{\sigma(P)}(\downarrow x \cap \downarrow D)$ is the Scott closure of the set $\downarrow D \cap \downarrow x$. In [11], Mao and Xu generalized the concept of meet continuous dcpos to general posets. Though meet continuous posets inherit some good properties of meet continuous dcpos, they fail to be *completion-invariant*. A simple counterexample will be given in Section 2.

In this paper, we introduce the concept of meet precontinuous posets. It is proved that a poset $P$ is meet precontinuous iff its normal completion is a meet continuous lattice; Considering the operator $\Gamma$ and the system $\gamma(P)$ in [2], we show that a poset $P$ is meet precontinuous iff the system $\gamma(P)$ is a complete Heyting algebra, and $P$ is a precontinuous poset and $\ll$ has the interpolation property iff $P$ is a meet precontinuous poset and for all $x \in P$, $U \in \gamma(P)^c$ which is the family of complements of elements of $\gamma(P)$, $x \in U$ implies that there are finite $F \subseteq P$ and $V \in \gamma(P)^c$ such that $x \in V \subseteq \uparrow F \subseteq U$ and $\Gamma$ is idempotent. We also locate the way below relation within auxiliary relations. It is proved that a poset $P$ is precontinuous iff the way below relation is the smallest approximating auxiliary relation iff $P$ is meet precontinuous and there is a smallest approximating auxiliary relation on $P$. Finally, we show how to construct all precontinuous posets whose way-below relation agrees with the given auxiliary relation.

## 2 Preliminaries

Let $P$ be a poset. For all $x \in P$, $A \subseteq P$, let $\uparrow x = \{y \in P : x \leq y\}$ and $\uparrow A = \bigcup_{a \in A} \uparrow a$; $\downarrow x$ and $\downarrow A$ are defined dually. Let $D(P) = \{A \subseteq P : A = \downarrow A\}$. $A^{\uparrow}$ and $A^{\downarrow}$ denote the sets of all upper and lower bounds of $A$, respectively. Let $A^{\delta} = (A^{\uparrow})^{\downarrow}$ and $\delta(P) = \{A^{\delta} : A \subseteq P\}$. $(\delta(P), \subseteq)$ is called the normal completion,