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$\mathbb{T}^\omega$ as a Stable Universal Domain [1]

Haoran Zhao, Hui Kou[2]

School of Mathematics
Sichuan University
Chengdu 610064, China

Abstract

In the seventies, G. Plotkin noticed that $\mathbb{T}^\omega$, the cartesian product of ω copies of the 3 elements flat domain of Boolean, is a universal domain, where "universal" means that the retracts of $\mathbb{T}^\omega$ in Scott's continuous semantics are exactly all the ωCC-domains, which with Scott continuous functions form a cartesian closed category. As usual "ω" is for "countably based", and here "CC" is for "conditionally complete", which essentially means that any subset which is pairwise bounded has an upper bound. Since $\mathbb{T}^\omega$ is also an ωDI-domain (an important structure in the stable domain theory), a problem arises naturally: Is $\mathbb{T}^\omega$ a universal domain for Berry's stable semantics? The aim of this paper is to answer this question. We investigate the properties of stable retracts and introduce a new domain named a conditionally complete DI-domain (a $CCDI$-domain for short). We show that, (1) a dcpo is a stable retract of $\mathbb{T}^\omega$ if and only if it is an $\omega CCDI$-domain; (2) the category of $\omega CCDI$-domain (resp. $CCDI$-domains) with stable functions is cartesian closed. So, the problem above has an affirmative answer.

Keywords: universal domain, stable retract, $\omega CCDI$-domain, cartesian closed category

1 Introduction

Domain theory is a general framework for defining program data domains. In this theory, $\mathbb{T}^\omega$, the cartesian product of ω copies of the 3 elements flat domain of Boolean, is a very interesting structure, which can be used as a model to give mathematical semantics for program languages as $P\omega$ presented by D. Scott [11]. In [10], G. Plotkin showed that, $\mathbb{T}^\omega$ is a universal domain in the sense that the retracts of $\mathbb{T}^\omega$ in Scott's continuous semantics form a cartesian closed category. Particularly, its continuous function space $[\mathbb{T}^\omega \to \mathbb{T}^\omega]$ is a retract of $\mathbb{T}^\omega$. R. Kanneganti [8] also investigated $\mathbb{T}^\omega$ in detail. The results of them are all based on the Scott continuous functions.

In domain theory, there is another class of important functions called stable functions, which is introduced firstly by Berry [4]. In 1990, P. Taylor [12] showed that,

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[2] Email: kouhui@scu.edu.cn

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all continuous (resp. algebraic) L-domains with stable functions form a cartesian closed category, its finite products are cartesian ones and its exponentials are stable function spaces. It leads many authors to study those categories for Berry's stable semantic, for example, see R. Amadio [2], Y.X. Chen and A. Jung [5], M. Droste [6], P-A Melliès [9], G.Q. Zhang [13,14], and so on. The theory based on stable functions is called the stable domain theory. In this theory, DI-domains are one of the most important class of stable domain structure. Each DI-domain is equivalent to a stable event structure and the category of DI-domains (resp. ωDI-domains) with stable functions is cartesian closed [13].

One see that $\mathbb{T}^\omega$ is also an ωDI-domain. So a problem arises naturally: Is $\mathbb{T}^\omega$ a stable universal domain in the sense that the category of all stable retracts of $\mathbb{T}^\omega$ with stable functions is cartesian closed? The aim of this paper is to answer this question. Since a stable retract is different to a continuous retract, we first investigate the properties of stable retracts. We introduce a new domain called a conditionally complete DI-domain and show that, (1) a dcpo is a stable retract of $\mathbb{T}^\omega$ if and only if it is an $\omega CCDI$-domain, where "ω" is for "countably based" as usual, and "CC" is for "conditionally complete", which essentially means that any subset which is pairwise bounded has an upper bound. i.e., a $CCDI$-domain with a countable base; (2) the category of $\omega CCDI$-domain (resp. $CCDI$-domains) with stable functions is cartesian closed. So, the problem above has an affirmative answer.

The paper is organized as follows. Section 1 is a introduction. Section 2 introduces some notions and definitions we need. Section 3 discusses the properties of stable retracts. Section 4 investigate the category of $\omega CCDI$-domain (resp. $CCDI$-domains). Section 5 investigate the stable retracts of $\mathbb{T}^\omega$. A pair of stable retract-stable embedding between $\mathbb{T}^\omega$ and an $\omega CCDI$-domain will be constructed in this section.

2 Preliminaries

We do assume some knowledge of basic domain theory, as in, e.g., [3,1,7]. A nonempty set P endowed with a partially order is called a poset. For $A \subseteq P$, we set $\downarrow A = \{x \in P : \exists a \in A,\ x \leq a\}$ and $\uparrow A = \{x \in P : \exists a \in P,\ a \leq x\}$, and A is called a lower or upper set, if $A = \downarrow A$ or $A = \uparrow A$ respectively. For an element $a \in P$, we use $\downarrow a$ or $\uparrow a$ instead of $\downarrow \{a\}$ or $\uparrow \{a\}$, and we say it a principal ideal or a principal filter, respectively. A subset D of P is called directed if it is nonempty and every nonempty finite subset of D has an upper bound in D. Particularly, we say that P is a dcpo if every directed subset D of P has a least upper bound (denoted by $\bigvee D$) in P.

For $x, y \in P$, we say that x is way-below y, denoted by $x \ll y$, if for any directed subset D of P, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$. P is continuous if $\{a \in P : a \ll x\}$ is directed and $x = \bigvee\{a \in P : a \ll x\}$ for all $x \in P$. A $k \in P$ is called compact if $k \ll k$. Let $K(P)$ be the set of all compact elements of P. P is called algebraic if $K(P) \cap \downarrow x$ is directed and $x = \bigvee(K(P) \cap \downarrow x)$ for all $x \in P$. A