Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science 300 (2014) 21–45

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Negative Modalities, Consistency and Determinedness

Adriano Dodó[2] João Marcos[3]

Department of Informatics and Applied Mathematics (DIMAp)
Federal University of Rio Grande do Norte (UFRN)
Natal, Brazil

Abstract

We study a modal language for negative operators —an intuitionistic-like negation and its paraconsistent dual—added to (bounded) distributive lattices. For each non-classical negation an extra operator is hereby adjoined in order to allow for standard logical inferences to be opportunely restored. We present abstract characterizations and exhibit the main properties of each kind of negative modality, as well as of the associated connectives that express consistency and determinedness at the object-language level. Appropriate sequent-style proof systems and adequate kripke semantics are also introduced, characterizing the minimal normal logic and a few other basic logics containing such negative modalities and their companions.

Keywords: Modal Logics, Paraconsistency, Paracompleteness, Derivability Adjustment.

1 Context

Negationless normal modal logics with box-like and diamond-like operators were studied by Dunn in [10], where the author obtains completeness results for the systems characterized by the class of all kripke frames and by a few specific subclasses thereof. In [7], Celani & Jansana extend that study so as to cover many other logics, and to that effect they consider kripke-style semantics based on frames containing two relations —one of them being a preorder, as in intuitionistic logic, allowing for the expression of appropriate heredity conditions. Systems containing analogous negative modalities were studied by Dunn & Zhou, who investigate in [11] modal logics with conjunction, disjunction, an impossibility operator intended to play the role of an intuitionistic-like negation and a non-necessity operator intended to play

[1] The research reported in this paper falls within the scope of the EU FP7 Marie Curie PIRSES-GA-2012-318986 project GeTFun: Generalizing Truth-Functionality. For their important feedback, the authors wish to thank in particular Cláudia Nalon, Carlos Caleiro, Arnon Avron, Ori Lahav and Yoni Zohar, as well as the audiences at IMLA/UNILOG and at SQIG/IT.

[2] Email: adrianoalvesdodo@gmail.com

[3] Email: jmarcos@dimap.ufrn.br

http://dx.doi.org/10.1016/j.entcs.2013.12.010

the role of a paraconsistent negation. Restall, in [16], proposed a combination of positive and negative diamond-like modal operators; one of his aims was to use the resulting system to exhibit examples of modal logics that turn out to be undecidable even in the absence of classical negation. In the present paper we will study a logic that contains the already mentioned negative modal operators over a strictly positive propositional basis (on a fragment agreed upon by intuitionists and classical logicians) to which we add extra operators that express at the object-language level the very notions of consistency and determinedness that allows one to recover much of the standard logical reasoning even when neither classical negation nor classical implication are available. The mentioned extra modal 'restoration' connectives were first proposed in [14]. The basic universal logic apparatus used here is based on [13] and [15], and the proof-theoretical approach to the consistency operator is inherited from [3].

The structure of the paper is as follows: in Section 2 we present the Universal Logic background, including the formulation of properties characterizing negative modalities, and the properties that characterize connectives intended to express consistency and determinedness at the object-language level; in Section 3 a sequent system is used to define our main and most basic modal system, in which we include rules for introducing the restoration connectives and rules for the interaction between the non-classical negations; in Sections 4 and 5 the intended kripke semantics is presented for our full modal language and our deductive system is shown to be sound and complete with respect to this semantics; a few extensions of the basic system are then formulated in Section 6; in Section 7 we study how the inferences of more standard logic systems may be recovered with the use of our rich modal language, by way of appropriate Derivability Adjustment Theorems; last, in Section 8, we briefly comment upon some directions for future research.

2 Universal Logic perspective

Let $\mathcal{L}$ be a standard propositional language. As customary, we shall use small Greek letters to denote arbitrary sentences, and capital Greek letters for sets of sentences of $\mathcal{L}$. A generalized consequence relation (gcr) will here be assumed to be a relation $\rhd \subseteq \mathbf{2}^{\mathcal{L}} \times \mathbf{2}^{\mathcal{L}}$ that enjoys the following universal properties:

(ovl) $\Gamma, \varphi \rhd \varphi, \Delta$

(mon) If $\Gamma_1 \rhd \Delta_1$, then $\Gamma_2, \Gamma_1 \rhd \Delta_1, \Delta_2$

(trn) If $\Gamma_1, \varphi \rhd \Delta_1$ and $\Gamma_2 \rhd \varphi, \Delta_2$, then $\Gamma_1, \Gamma_2 \rhd \Delta_1, \Delta_2$

In writing a statement such as $\Pi \cup \{\pi\} \rhd \varnothing$ in the simplified form $\Pi, \pi \rhd$ we are simply aligning with standard usage from the literature. Here we shall write $\Gamma \blacktriangleright \Delta$ to indicate that $\Gamma \rhd \Delta$ fails, that is, that $\langle \Gamma, \Delta \rangle \notin \rhd$. Furthermore, aiming at a structured outlook on the above properties and on proofs based on them, we shall employ the following graphical representation: